GRAVITATION.

GRAVITATION:

AN

ELEMENTARY EXPLANATION

OF

THE PRINCIPAL PERTURBATIONS

IN THE

SOLAR SYSTEM.

(WRITTEN FOR THE PENNY CYCLOPÆDIA, AND NOW PREVIOUSLY PUBLISHED FOR THE USE OF STUDENTS IN THE UNIVERSITY OF CAMBRIDGE.)

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PREFACE.

In laying this work before the public, I think it right to state the object for which it was originally composed, and the circumstances which have in some degree changed its destination.

The treatise was originally designed for a class of readers who might be supposed to possess a moderate acquaintance with the phænomena and the terms of astronomy; geometrical notions sufficient to enable them to understand simple inferences from diagrams; two or three terms of algebra as applied to numbers; but none of that elevated science which has always been used in the investigation of these subjects, and without which scarcely an attempt has been made to explain them. I proposed to myself, therefore, this general design: to explain the perturbations of the solar system, as far as I was able, without introducing an algebraic symbol.

It will readily be believed that, after thus denying myself the use of the most powerful engine of mathematics, I did not expect to proceed very far. In my progress, however, I was surprised to find that a general explanation, perfectly satisfactory, might be offered for almost every inequality recognised as sensible in works on Physical Astronomy. I now began to conceive it possible that the work, without in the smallest degree departing from the original plan, or giving up the original object, might also be found useful to a body of students, furnished with considerable mathematical powers, and in the habit of applying them to the explanation of difficult physical problems. With this idea, the treatise is now printed in a separate form.

The utility of a popular explanation of profound physical investigations is not, in my opinion, to be restricted to the instruction of readers who are unable to pursue them with the powers of modern analysis. Much is done when the interest of a good mathematician is excited by seeing, in a form that can be easily understood, results which are important for the comprehension of the system of the universe, and which can be made complete only by the application of a higher calculus. That such an interest has operated powerfully in our Universities, I have no doubt. How many of our students would have known any thing of the Lunar Theory, if they had not been enjoined to read Newton's eleventh section? And how many at this time possess the least acquaintance with the curious and complicated, but beautiful, theory of Jupiter's satellites, of which no elementary explanation is laid before them? But this is not all. The exercise of the mind in understanding a series of propositions, where the last conclusion is geometrically in close connexion with the first cause, is very different from that which it receives from putting in play the long train of machinery in a profound analytical process. The degrees of conviction in the two cases are very different. It is known to every one who has been engaged in the instruction of students at our Universities, that the results of the differential calculus are received by many, rather with the doubts of imperfect faith than with the confidence of rational conviction. Nor is this to be wondered at; a clear understanding of many difficult steps, a distinct perception that every connexion of these steps is correct, and a general comprehension of the relations of the whole series of steps, are necessary for complete confidence. An unusual combination of talents, attainments, and labour, must be required, to appreciate clearly the evidence for a result of deep analysis. I am not unwilling to avow that the simple considerations which have been forced upon me in the composition of this treatise, have, in several instances, contributed much to clear up my view of points, which before were obscure, and almost doubtful. To the greater number of students, therefore, I conceive a popular geometrical explanation is more useful than an algebraic investigation. But even to those who are able to pursue the investigations with a skilful use of the most powerful methods, I imagine that a popular explanation is not unserviceable. The insight which it gives into the relation of some mechanical causes and geometrical effects, may powerfully, yet imperceptibly, influence their understanding of many others which occur in the prosecution of an algebraical process. The advanced student who exults in the progress which the modern calculus enables him to make in the Lunar or Planetary Theories, perhaps, hardly reflects how much of the power of understanding his conclusions has been derived from Newton's general explanations.

The utility of such a work being allowed, it cannot, I think, be disputed that there exists a necessity for a new one. The only attempts at popular explanation in general use with which I am acquainted, are Newton's eleventh section, and a small part of Sir John Herschel's admirable treatise on Astronomy. The former of these (the most valuable chapter that has ever been written on physical science,) is in some parts very defective. Thus, the explanation of the motion of the line of apses is too general, and enters into particular cases too little, to allow of a numerical calculation being founded on it. The explanation of evection is extremely defective. The explanation of variation, however, and of alteration of the node and inclination, are probably as complete as can be given. The latter treatise, besides expanding some of Newton's reasoning, alludes to the long inequalities and secular disturbances of the planets, but not perhaps with sufficient accuracy of detail to supersede the necessity of further explanation. No popular work with which I am acquainted, alludes at all to the peculiarities of the theory of Jupiter's satellites.

I have attempted in some degree to supply these defects; with what success the reader must judge. As it was my object to avoid repetition of theorems, which are to be found in treatises on Mechanics and elementary works on Physical Astronomy, and which are fully read and mastered by those who take much interest in these subjects, and which, moreover, do not admit of popular explanation so easily as many of the more advanced propositions, I have omitted noticing them any further than the consistency of system seemed to require. Thus, with regard to elliptic motion, Kepler's laws, &c., I have merely stated results; because the investigations of these are familiar to the higher students, to whom I hope the other explanations may be useful; and because without great trouble it did not appear possible to put the reasons for these results in the same form as those for other effects of force. I have, however, alluded to some of the difficulties which are apt to embarrass readers in the first instance, as much for the sake of the reasoning contained in the explanation as for the value of the results. The only additions which I have thought it desirable to make for the benefit of readers of Newton, are contained in a few notes referring to one of Newton's constructions.

To the reader who may detect faults in the composition of the work, I can merely state in apology, that it has been written in a hurried manner, in the intervals of very pressing employments. I have only to add, that, holding a responsible situation in my University, I have always thought it my duty to promote, as far as I am able, the study of Physical Astronomy; and that if this treatise shall contribute to extend the knowledge of its phænomena and their relation to their causes, either among the students of the University, or in that more numerous body for whom it was originally written, I shall hold myself well repaid for the trouble which it has cost me.

G. B. Airy.

Observatory, Cambridge,
March 9, 1834.

CONTENTS.

ON GRAVITATION.

SECTION I.—*On the Rules for calculating Attraction, or, the Law of Gravitation.*

(1.) THE principle upon which the motions of the earth, moon, and planets are calculated is this: Every particle of matter *attracts* every other particle. That is, if there were a single body alone, and at rest, then, if a second body were brought near it, the first body would immediately begin to move toward the second body. Just in the same manner, if a needle is at rest on a table, and if a magnet is brought near it, the needle immediately begins to move towards the magnet, and we say that the magnet attracts the needle. But magnetic attraction belongs only to certain bodies: whereas the attraction of which we speak here belongs to all bodies of every kind: metals, earths, fluids, and even the air and gases are equally subject to its influence.

(2.) The most remarkable experiments which prove that bodies attract each other are a set of experiments made at the end of the last century by Mr. Cavendish. Small leaden balls were supported on the ends of a rod which was suspended at the middle by a slender wire; and when large leaden balls were brought near to them, it was found that the wire was immediately twisted by the motion of the balls. But the results of this experiment are striking, principally because they are unusual; the ordinary force of gravity serves quite as well to prove the existence of some such power. For when we consider that the earth is round, and that, on all parts of it, bodies, as soon as they are at liberty, fall in directions perpendicular to its surface, (and therefore fall in opposite directions at the places which are diametrically opposite,) we are compelled to allow that there is a force such as we call attraction, either directed to the centre of the earth, or produced by a great number of small forces, directed to all the different particles composing the earth. The peculiar value of Cavendish's experiment consists in showing that there is a small force directed to every different particle of the earth.

(3.) But it is necessary to state distinctly the rules by which this attraction is regulated, and by which it may be calculated; or (as it is technically called) *the law of gravitation.* Before we can do this, we must determine which of the effects of attraction we choose to take as its measure. For there are two distinct effects: one is the *pressure* which it produces upon any obstacle that keeps the body at rest; the other is the *space through which it draws the body in a certain time,* if the obstacle is removed and the body set at liberty. Thus, to take the ordinary force of gravity as an instance: we might measure it by the pressure which is produced on the hand by a lump of lead held in the hand; or we might measure it by the number of inches through which the lump of lead would fall in a second of time after the hand is opened (as the pressure and the fall are both occasioned by gravity). But there is this difference between the two measures; if we adopted the first, since a large lump of lead weighs more than a small one, we should find a different measure by the use of every different piece of lead; whereas, if we adopt the second, since it is well established by careful and accurate experiments that large and small lumps of lead, stones, and even feathers, fall through the same number of inches in a second of time, (when the resistance of the air, &c., is removed,) we shall get the same measure for gravity, whatever body we suppose subject to its influence. The consistence and simplicity of the measure thus obtained incline us to adopt it in every other case; and thus we shall say, *Attraction is measured by the space through which it draws a body in one second of time after the body is set at liberty.*

(4.) Whenever we speak, therefore, of calculating attraction, it must be understood to mean calculating the number of inches, or feet, through which the attraction draws a body in one second of time.

(5.) Now the first rule is this : "The attraction of one body upon another body does not depend on the mass of the body which is attracted, but is the same whatever be the mass of the body so attracted, if the distances are the same."

(6.) Thus Jupiter attracts the sun, and Jupiter attracts the earth also; but though the sun's mass is three hundred thousand times as great as the earth's, yet the attraction of Jupiter on the sun is exactly equal to his attraction on the earth, when the sun and the earth are equally distant from Jupiter. In other words, (the attraction being measured in conformity with the definition above,) when the sun and the earth are at equal distances from Jupiter, the attraction of Jupiter draws the sun through as many inches, or parts of an inch, in one second of time as it draws the earth in the same time.

(7.) The second rule is this: "Attraction is proportional to the mass of the body which attracts, if the distances of different attracting bodies be the same."

(8.) Thus, suppose that the sun and Jupiter are at equal distances from Saturn; the sun is about a thousand times as big as Jupiter; then whatever be the number of inches through which Jupiter draws Saturn in one second of time, the sun draws Saturn in the same time through a thousand times that number of inches.

(9.) The third rule is this: "If the same attracting body act upon several bodies at different distances, the attractions are inversely proportional to the square of the distances from the attracting body."

(10.) Thus the earth attracts the sun, and the earth also attracts the moon; but the sun is four hundred times as far off as the moon, and therefore, the earth's attraction on the sun is only $\frac{1}{160000}$th part of its attraction on the moon; or, as the earth's attraction draws the moon through about $\frac{1}{20}$th of an inch in one second of time, the earth's attraction draws the sun through $\frac{1}{3200000}$th of an inch in one second of time. In like manner, supposing Saturn ten times as far from the sun as the earth is, the sun's attraction upon Saturn is only one hundredth part of his attraction on the earth.

(11.) The same rule holds in comparing the attractions which one body exerts upon another, when, from moving in different paths, and with different degrees of swiftness, their distance is altered. Thus Mars, in the spring of 1833, was twice as far from the earth as in the autumn of 1832; therefore, in the spring of 1833, the earth's attraction on Mars was only one-fourth of its attraction on Mars in the autumn of 1832. Jupiter is three times as near to Saturn, when they are on the same side of the sun as when they are on opposite sides; therefore, Jupiter's attraction on Saturn, and Saturn's attraction on Jupiter, are nine times greater when they are on the same side of the sun than when they are on opposite sides.

(12.) The reader may ask, How is all this known to be true? The best answer is, perhaps, the following: We find that the force which the earth exerts upon the moon bears the same proportion to gravity on the earth's surface, which it ought to bear in conformity with the rule just given. For the motions of the planets, calculations are made, which are founded upon these laws, and which will enable us to predict their places with considerable accuracy, if the laws are true, but which would be much in error if the laws were false. The accuracy of astronomical observations is carried to a degree that can scarcely be imagined; and by means of these we can every day compare the observed place of a planet with the place which was calculated beforehand, according to the law of gravitation. It is found that they agree so nearly, as to leave no doubt of the truth of the law. The motion of Jupiter, for instance, is so perfectly calculated, that astronomers have computed ten years beforehand the time at which it will pass the meridian of different places, and we find the predicted time correct within half a second of time.

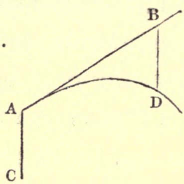
Section II.—*On the Effect of Attraction upon a Body which is in motion, and on the Orbital Revolutions of Planets and Satellites.*

(13.) We have spoken of the simplest effects of attraction, namely, the production of pressure, if the matter on which the attraction acts is supported, (as when a stone is held in the hand,) and the production of motion if the matter is set at liberty, (as when a stone is dropped from the hand.) And it will easily be understood, that when a body is projected, or thrown, in the same direction in which the force draws it, (as when a stone is thrown downwards,) it will move with a greater velocity than either of these causes separately would have given it; and if thrown in the direction opposite to that in which the force draws it, (as when a stone is thrown upwards,) its motion will become slower and slower, and will, at last, be turned into a motion in the opposite direction. We have yet to consider a case much more important for astronomy than either of these: Suppose that a body is projected in a direction *transverse to,* or *crossing,* the direction in which the force draws it, how will it move?

(14.) The simplest instance of this motion that we can imagine is the motion of a stone when it is thrown from the hand in a horizontal direction, or in a direction nearly horizontal. We all know that the stone soon falls to the ground; and if we observe its motion with the least attention, we see that it does not move in a straight line; it begins to move in the direction in which it is thrown; but this direction is speedily changed; it continues to change gradually and constantly, and the stone strikes the ground, moving at that time in a direction much inclined to the original direction. The most powerful effort that we can make, even when we use artificial means, (as in producing the motion of a bomb or a cannon-ball,) is not sufficient to prevent the body from falling at last. This experiment, therefore, will not enable us immediately to judge what will become of a body (as a planet) which is put in motion at a great distance from another body, which attracts it, (as the sun;) but it will assist us much in judging generally what is the nature of motion when a body is projected in a direction transverse to the direction in which the force acts on it.

(15.) It appears, then, that the general nature of the motion is this: the body describes a curved path, of which the first part has the same direction as the line in which it is projected. The circumstances of the motion of the stone may be calculated with the utmost accuracy from the following rule, called the second law of motion, (the accuracy of which has been established by many simple experiments, and many inferences from complicated motion.) If A, *fig.* 1, is the point from which the stone was thrown, and A B the

Fig. 1.



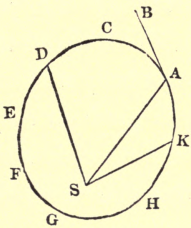
direction in which it was thrown; and if we wish to know where the stone will be at the end of any particular time, (suppose, for instance, three seconds,) and if the velocity with which it is thrown would, in three seconds, have carried it to B, supposing gravity not to have acted on it; and if gravity would have made it fall from A to C, supposing it to have been merely dropped from the hand; then, at the end of three seconds, the stone really will be at the point D, which is determined by drawing B D parallel and equal to A C; and it will have reached it by a curved path A D, of which different points can be determined in the same way for different instants of time.

(16.) The calculation of the stone's course is easy, because, during the whole motion of the stone, gravity is acting upon it with the same force and in the same direction. The circumstances of the motion of a body attracted by a planet, or by the sun, (where the force, as we have before mentioned, is inversely proportional to the square of the distance, and therefore varies as the distance alters, and is not the same, either in its amount or in its direction at the point D, as it is at the point C,) cannot be computed by the same simple method. But the same method will apply, provided we restrict the intervals for which we make the calculations to times so short, that the alterations in the amount of the force, and in its direction, during each of those times, will be very small. Thus, in the motion of the earth, as affected by the attraction of the sun, if we used the process that we have described, to find where the earth will be at the end of a month from the present time, the place that we should find would be very far wrong; if we calculated for the end of a week, since the direction of the force (always directed to the sun) and its magnitude (always proportional inversely to the square of the distance from the sun) would have been less altered, the circumstances would have been more similar to those of the motion of the stone, and the error in the place that we should find would be much less than before; if we calculated by this rule for the end of a day, the error would be so small as to be perceptible only in the nicest observations; and if we calculated for the end of a minute, the error would be perfectly insensible.

(17.) Now a method of calculation has been invented, which amounts to the same as making this computation for every successive small portion of time, with the correct value of the attractive force, and the correct direction of force at every particular portion of time, and finding thus the place where the body will be at the end of any time that we may please to fix on, without the smallest error. The rules to which this leads are simple: but the demonstration of the rules requires the artifices of advanced science. We cannot here attempt to give any steps of this demonstration; but our plan requires us to give the results.

(18.) It is demonstrated that if a body (a planet, for instance) is by some force projected from A, *fig.* 2, in the direction A B, and if the attraction of the sun, situated at S, begins immediately to act on

Fig. 2.



it, and continues to act on it according to the law that we have mentioned, (that is, being inversely proportional to the square of its distance from S, and always directed to S;) and if no other force whatever but this attraction acts upon the body; then the body will move in one of the following curves—a circle, an ellipse, a parabola, or a hyperbola.

In every case the curve will, at the point A, have the same direction as the line A B; or, (to use the language of mathematicians,) A B will be a tangent to the curve at A.

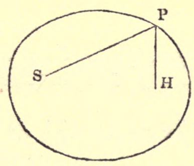
The curve cannot be a circle unless the line A B is perpendicular to S A, and, moreover, unless the velocity, with which the planet is projected, is neither greater nor less than one particular velocity determined by the length of S A and the mass of the body S. If it differs little from this particular velocity, (either greater or less,) the body will move in an ellipse; but if it is much greater, the body will move in a parabola or a hyperbola.

If A B is oblique to S A, and the velocity of projection is small, the body will move in an ellipse; but if the velocity is great, it may move in a parabola or hyperbola, but not in a circle.

If the body describe a circle, the sun is the centre of the circle.

If the body describe an ellipse, the sun is not the centre of the ellipse, but one focus. (The method of describing an ellipse is to fix two pins in a board, as at S and H, *fig.* 3; to fasten a thread

Fig. 3.



S P H to them, and to keep this thread stretched by the point of a pencil, as at P: the pencil will trace out an ellipse, and the places of the pins S and H will be the two focuses.)

If the body describe a parabola or hyperbola, the sun is in the focus.

(19.) The planets describe ellipses which are very little flattened, and differ very little from circles.

Three or four comets describe very long ellipses: and nearly all the others that have been observed are found to move in curves which cannot be distinguished from parabolas. There is reason to think that two or three comets which have been observed move in hyperbolas. But as we do not propose, in this treatise, to enter into a discussion on the motions of comets, we shall confine ourselves to the consideration of motion in an ellipse.

(20.) Every thing that has been said respecting the motion of a planet, or body of any kind, round the sun, in consequence of the sun's attraction according to the law of gravitation, applies equally well to the motion of a satellite about a planet, since the planet attracts with a force following the same law (though smaller) as the attraction of the sun. Thus the moon describes an ellipse round the earth, the earth being the focus of the ellipse; Jupiter's satellites describe each an ellipse about Jupiter, and Jupiter is in one focus of each of those ellipses; the same is true of the satellites of Saturn and Uranus.

(21.) In stating the suppositions on which the calculations of orbits are made, we have spoken of a force of attraction, and a force by which a planet is projected. But the reader must observe that the nature of these forces is wholly different. The force of attraction is one which acts constantly and steadily without a moment's intermission, (as we know that gravity to the earth is always acting :) the force by which the body is projected is one which we suppose to be necessary at some past time to account for the planet's motion, but which acts no more. The planets *are in motion,* and it is of no consequence to our inquiry how they received this motion, but it is convenient, for the purposes of calculation, to suppose that, at some time, they received an impulse of the same kind as that which a stone receives when thrown from the hand; and this is the whole meaning of the term "projectile force."

(22.) From the same considerations it will appear that, if in any future investigations we should wish to ascertain what is the orbit described by a planet after it leaves a certain point where the velocity and direction of its motion are known, we may suppose the planet to be projected from that point with that velocity and in that direction. For it is unimportant by what means the planet acquires its velocity, provided it has such a velocity there.

(23.) We shall now allude to one of the points which, upon a cursory view, has always appeared one of the greatest difficulties in the theory of elliptic revolution, but which, when duly considered, will be found to be one of the most simple and natural consequences of the law of gravitation.

(24.) The force of attraction, we have said, is inversely proportional to the square of the distance, and is therefore greatest when the distance is least. It would seem then, at first sight, that when a planet has approached most nearly to the sun, as the sun's attraction is then greater than at any other time, the planet must inevitably fall to the sun. But we assert that the planet begins then to recede from the sun, and that it attains at length as great a distance as before, and goes on continually retracing the same orbit. How is this receding from the sun to be accounted for?

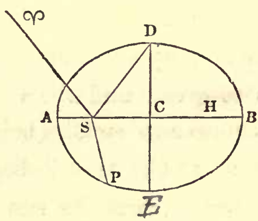
(25.) The explanation depends on the increase of velocity as the planet approaches to the point where its distance from the sun is least, and on the considerations by which we determine the form of the curve which a certain attracting force will cause a planet to describe. In explaining the motion of a stone thrown from the hand, to which the motion of a planet for a very small time is exactly similar, we have seen that the deflection of the stone from the straight line in which it began to move is exactly equal to the space through which gravity could have made it fall in the same time from rest, whatever were the velocity with which it was thrown. Consequently, when the stone is thrown with very great velocity, it will have gone a great distance before it is much deflected from the straight line, and therefore its path will be very little curved; a fact familiar to the experience of every one. The same thing holds with regard to the motion of a planet, and thus the curvature of any part of the orbit which a planet describes will not depend simply upon the force of the sun's attraction, but will also depend on the velocity with which the planet is moving. The greater is the velocity of the planet at any point of its orbit, the less will the orbit be curved at that part. Now if we refer to *fig.* 2, we shall see that, supposing the planet to have passed the point C with so small a velocity that the attraction of the sun bends its path very much, and causes it immediately to begin to approach towards the sun; the sun's attraction will necessarily increase its velocity as it moves through D, E, and F. For the sun's attractive force on the planet, when the planet is at D, is acting in the direction D S, and it is plain that (on account of the small inclination of D E to D S) the force pulling in the direction D S, helps the planet along in its path D E, and thereby increases its velocity. Just as when a ball rolls down a sloping bank, the force of gravity (whose direction is not much inclined to the bank) helps the ball down the bank, and thereby increases its velocity. In this manner, the velocity of the planet will be continually increasing as the planet passes through D, E, and F; and though the sun's attractive force (on account of the planet's nearness) is very much increased, and tends, therefore, to make the orbit more curved, yet the velocity is so much increased that, on that account, the orbit is not more curved than before. Upon making the calculation more accurately, it is found that the planet, after leaving C, approaches to the sun more and more rapidly for about a quarter of its time of revolution; then for about a quarter of its time of revolution the velocity of its approach is constantly diminishing: and at half the periodic time after leaving C, the planet is no longer approaching to the sun; and its velocity is so great, and the curvature of the orbit in consequence so small, (being, in fact, exactly the same as at C,) that it begins to recede. After this it recedes from the sun by exactly the same degrees by which it before approached it.

(26.) The same sort of reasoning will show why, when the planet reaches its greatest distance, where the sun's attraction is least, it does not altogether fly off. As the planet passes along H, K, A, the sun's attraction (which is always directed to the sun) retards the planet in its orbit, just as the force of gravity retards a ball which is bowled up a hill; and when it has reached C, its velocity is extremely small; and, therefore, though the sun's attraction at C is small, yet the deflection which it produces in the planet's motion is (on account of the planet's slowness there) sufficient to make its path very much curved, and the planet approaches the sun, and goes over the same orbit as before.

(27.) The following terms will occur perpetually in the rest of this treatise, and it is therefore desirable to explain them now.

Let S and H, *fig.* 4, be the focuses of the ellipse

Fig. 4.



AEDB; draw the line AB through S and H;

take C the middle point between S and H, and draw D C E perpendicular to A C B. Let S be that focus which is the place of the sun, (if we are speaking of a planet's orbit,) or the place of the planet (if we are speaking of a satellite's orbit.)

Then A B is called the *major axis* of the ellipse.

C is the *centre.*

A C or C B is the *semi-major axis.* This is equal in length to S D; it is sometimes called the *mean distance,* because it is half-way between A S (which is the planet's smallest distance from S) and B S, (which is the planet's greatest distance from S.)

DE is the *minor axis,* and D C or C E the *semi-minor axis.*

A is called the *perihelion,* (if we are speaking of a planet's orbit;) the *perigee,* (if we are speaking of the orbit described by our moon about the earth;) the *perijove,* (if we are speaking of the orbit described by one of Jupiter's satellites round Jupiter;) or the *perisaturnium,* (if we are speaking of the orbit described by one of Saturn's satellites about Saturn.)

B, in the orbit of a planet, is called the *aphelion;* in the moon's orbit it is called the *apogee;* in the orbit of one of Jupiter's satellites, we shall call it the *apojove.*

A and B are both called *apses;* and the line A B, or the major axis, is sometimes called the *line of apses.*

S C is sometimes called the *linear excentricity;* but it is more usual to speak only of the proportion which S C bears to A C, and this proportion, expressed by a number, is called the *excentricity.* Thus, if S C were one-third of A C, we should say, that the excentricity of the orbit was $\frac{1}{3}$, or 0·3333.

If S ♈ is drawn towards a certain point in the heavens, called the *first point of Aries,* then the angle ♈ S A is called the *longitude of perihelion,* (or of perigee, or of perijove, &c.)

If P is the place of the planet in its orbit at any particular time, then the angle ♈ S P is its *longitude* at that time, and the angle A S P is its *true anomaly.* (The longitude of the planet is, therefore, equal to the sum of the longitude of the perihelion, and the true anomaly of the planet.) The line S P is called the *radius vector.*

In all our diagrams it is to be understood, that the planet, or satellite, moves through its orbit in the direction opposite to the motion of the hands of a watch. This is the direction in which all the planets and satellites would appear to move, if viewed from any place on the north side of the planes of their orbits.

The time in which the planet moves from any one point of the orbit through the whole orbit, till it comes to the same point again, is called the planet's *periodic time.*

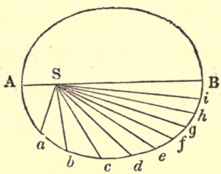
(28.) If we know the mass of the central body, and if we suppose the revolving body to be projected at a certain place in a known direction with a given velocity, the length of the axis major, the excentricity, the position of the line of apses, and the periodic time, may all be calculated. We cannot point out the methods and formulæ used for these, but we may mention one very remarkable result. The length of the axis major depends only upon the velocity of projection, and upon the place of projection, and not at all upon the direction of projection.

(29.) We shall proceed to notice the principle on which the motion of a planet, or satellite, in its orbit is calculated.

It is plain that this is not a very easy business. We have already explained, that the velocity of the planet in its orbit is not uniform, (being greatest when the planet's distance from the sun is least, or when the planet is at perihelion;) and it is obvious, that the longitude of the planet increases very irregularly; since, when the planet is near to the sun, its actual motion is very rapid, and, therefore, the increase of longitude is extremely rapid; and when the planet is far from the sun, its actual motion is slow, and, therefore, the increase of longitude is extremely slow. The rule which is demonstrated by theory, and which is found to apply precisely in observation, is this:—The areas described by the radius vector are equal in equal times. This is true, whether the force be inversely as the square of the distance from the central body, or be in any other proportion, provided that it is directed to the central body.

(30.) Thus, if in one day a planet, or a satellite, moves from A to *a*, *fig.* 5; in the next day it will

Fig. 5.



move from *a* to *b*, making the area *a* S *b* equal to A S *a*; in the third day it will move from *b* to *c*, making the area *b* S *c* equal to A S *a* or *a* S *b*, and so on.

(31.) Upon this principle mathematicians have invented methods of calculating the place of a planet, or satellite, at any time for which it may be required. These methods are too troublesome for us to explain here; but we may point out the meaning of two terms which are frequently used in these computations. Suppose, for instance, as in the figure, that the planet, or satellite, occupies ten days in describing the half of its orbit, A*a b c d e f g h i* B, or twenty days in describing the whole orbit; and suppose that we wished to find its place at the end of three days after leaving the perihelion. If the orbit were a circle, the planet would in three days have moved through an angle of 54 degrees. If the excentricity of the orbit were small, (that is, if the orbit did not differ much from a circle,) the angle through which the planet would have moved would not differ much from fifty-four degrees. The excentricities of all the orbits of the planets are small; and it is convenient, therefore, to begin with the angle 54° as one which is not very erroneous, but which will require some correction. This angle (as 54°), which is proportional to the time, is called the *mean anomaly;* and the correction which it requires, in order to produce the true anomaly, is called the *equation of the centre.* If we examine the nature of the motion, while the planet moves from A to B, it will readily be seen, that, during the whole of that time, the angle really described by the planet is *greater* than the angle which is proportional to the time, or the equation of the centre is to be *added to* the mean anomaly, in order to produce the true anomaly; but while the planet moves in the other half of the orbit, from B to A, the angle really described by the planet is *less* than the angle which is proportional to the time, or the equation of the centre is to be *subtracted from* the mean anomaly, in order to produce the true anomaly.

(32.) The sum of the mean anomaly and the longitude of perihelion is called the *mean longitude* of the planet. It is evident, that if we add the equation of the centre to the mean longitude, while the planet is moving from A to B, or subtract it from the mean longitude, while the planet is moving from B to A, as in (31.), we shall form the true longitude.

(33.) The reader will see, that when the planet's true anomaly is calculated, the length of the radius vector can be computed from a knowledge of the properties of the ellipse. Thus the place of the planet, for any time, is perfectly known. This problem has acquired considerable celebrity under the name of Kepler's problem.

(34.) There remains only one point to be explained regarding the undisturbed motion of planets and satellites; namely, the relation between a planet's periodic time and the dimensions of the orbit in which it moves.

Now, on the law of gravitation it has been demonstrated from theory, and it is fully confirmed by observation, that the periodic time does not depend on the excentricity, or on the perihelion distance, or on the aphelion distance, or on any element except the *mean distance* or semi-major axis. So that if two planets moved round the sun, one in a circle, or in an orbit nearly circular, and the other in a very flat ellipse; provided their mean distances were equal, their periodic times would be equal. It is demonstrated also, that for planets at different distances, the relation between the periodic times and the mean distances is the following:—The squares of the numbers of days (or hours, or minutes, &c.) in the periodic times have the same proportion as the cubes of the numbers of miles, (or feet, &c.) in the mean distances.

(35.) Thus the periodic time of Jupiter round the sun is 4332·7 days, and that of Saturn is 10759·2 days; the squares of these numbers are 18772289 and 115760385. The mean distance of

Jupiter from the sun is about 487491000 miles, and that of Saturn is about 893955000 miles; the cubes of these numbers are 1158496 (20 ciphers), and 7144088 (20 ciphers). On trial it will be found, that 18772289 and 115760385 are in almost exactly the same proportion as 1158496 and 7144088.

(36.) In like manner, the periodic times of Jupiter's third and fourth satellites round Jupiter are 7·15455 and 16·68877 days; the squares of these numbers are 51·1876 and 278·515. Their mean distances from Jupiter are 670080 and 1178560 miles; the cubes of these numbers are 300866 (12 ciphers), and 1637029 (12 ciphers), and the proportion of 51·1876 to 278·515 is almost exactly the same as the proportion of 300866 to 1637029.

(37.) It must, however, be observed that this rule applies in comparing the periodic times and mean distances, *only* of bodies which revolve round the *same* central body. Thus the rules applies in comparing the periodic times and mean distances of Jupiter and Saturn, because they both revolve round the sun; it applies in comparing the periodic times and mean distances of Jupiter's third and fourth satellites, because they both revolve round Jupiter; but it would not apply in comparing the periodic time and mean distance of Saturn revolving round the sun with that of Jupiter's third satellite revolving round Jupiter.

(38.) In comparing the orbits described by different planets, or satellites, round different centres of force, theory gives us the following law:—The cubes of the mean distances are in the same proportion as the products of the mass by the square of the periodic time. Thus, for instance, the mean distance of Jupiter's fourth satellite from Jupiter is 1178560 miles; its periodic time round Jupiter is 16·68877 days; the mean distance of the earth from the sun is 93726900 miles; its periodic time round the sun is 365·2564 days; also the mass of Jupiter is $\frac{1}{1050}$th the sun's mass. The cubes of the mean distances are respectively 1637029 (12 ciphers), and 823365 (18 ciphers); the products of the squares of the times by the masses are respectively 0·265252 and 133412; and these numbers are in the same proportion as 1637029 (12 ciphers), and 823365 (18 ciphers).

(39.) The three rules, that planets move in ellipses; that the radius vector in each orbit passes over areas proportional to the times, and that the squares of the periodic times are proportional to the cubes of the mean distances, are commonly called *Kepler's laws.* They were discovered by Kepler from observation, before the theory of gravitation was invented; they were first explained from the theory by Newton, about A.D. 1680.

(40.) The last of these is not strictly true, unless we suppose that the central body is absolutely immoveable. This, however, is evidently inconsistent with the principles which we have laid down in Section I. In considering the motion, for instance, of Jupiter round the sun, it is necessary to consider, that, while the sun attracts Jupiter, Jupiter is also attracting the sun. But the planets are so small in comparison with the sun, (the largest of them, Jupiter, having less than one-thousandth part of the matter contained in the sun,) that in common illustrations there is no need to take this consideration into account. For nice astronomical purposes it is taken into account in the following manner:—The motion which the attraction of Jupiter produces in the sun is less than the motion which the attraction of the sun produces in Jupiter, in the same proportion in which Jupiter is smaller than the sun. If the sun and Jupiter were allowed to approach one another, their rate of approach would be the *sum* of the motions of the sun and Jupiter, and would, therefore, be greater than their rate of approach, if the sun were not moveable, in the same proportion in which the sum of the masses of the sun and Jupiter is greater than the sun's mass. That is, the rate of approach of the sun and Jupiter, both being free, is the same as the rate of approach would be if the sun were fixed, provided the sun's mass were increased by adding Jupiter's mass to it. Consequently, in comparing the orbits described by different planets round the sun, we must use the rule just laid down, supposing the central force to be the attraction of a mass equal to the sum of the sun and the planet; and thus we get a proportion which is rigorously true: for different planets, or even for different bodies, revolving round different centres of force, the cubes of the mean distances are in the same proportion as the products of the square of the periodic time by the sum of the masses of the attracting and attracted body.

Section III. *General Notions of Perturbation; and Perturbation of the Elements of Orbits.*

(41.) We have spoken of the motion of two bodies (as the sun and a planet) as if no other attracting body existed. But, as we have stated in Section I., every planet and every satellite attracts the sun and every other planet and satellite. It is plain now that, as each planet is attracted very differently at different times by the other planets whose position is perpetually varying, the motion is no longer the same as if it was only attracted by the sun. The planets, therefore, do not move exactly in ellipses; the radius vector of each planet does not pass over areas exactly proportional to the times; and the proportion of the cube of the mean distance to the product of the square of the periodic time by the sum of the masses of the sun and the planet, is not strictly the same for all. Still the disturbing forces of the other planets are so small in comparison with the attraction of the sun, that these laws are very nearly true; and (except for our moon and the other satellites) it is only by accurate observation, continued for some years, that the effects of perturbation can be made sensible.

(42.) The investigation of the effects of the disturbing forces will consist of two parts: the examination into the effects of disturbing forces generally upon the motion of a planet, and the examination into the kind of disturbing force which the attraction of another planet produces. We shall commence with the former; we shall suppose that a planet is revolving round the sun, the sun being fixed, (a supposition made only for present convenience,) and that some force acts on the planet without acting on the sun, (a restriction introduced only for convenience, and which we shall hereafter get rid of.)

(43.) The principle upon which we shall explain the effect of this force is that known to mathematicians by the name of *variation of elements.* The planet, as we have said, describes some curve which is not strictly an ellipse, or, indeed, any regularly formed curve. It will not even describe the same curve in successive revolutions. Yet its motion may be represented by supposing it to have moved in an ellipse, provided we suppose the elements of the ellipse to have been perpetually altering. It is plain that by this contrivance any motion whatever may be represented. By altering the major axis, the excentricity, and the longitude of perihelion, we may in many different ways make an ellipse that will pass through any place of the planet; and by altering them in some particular proportions, we may, in several ways, make an ellipse in which the direction of motion at the place of the planet shall be the same as the direction of the planet's motion. But there is only one ellipse which will pass exactly through a place of the planet, in which the direction of the motion at that place shall be exactly the same as the direction of the planet's motion, and in which the velocity (in order that a body may revolve in that ellipse round the sun) will be the same as the planet's real velocity. The dimensions and position of this ellipse may be conceived as follows: if at any instant we suppose the disturbing force to cease, and conceive the planet to be as it were projected with the velocity which it happens to have at that instant, the attraction of the sun or central body will cause it to describe the ellipse of which we are speaking. We shall in future mention this by the name of the *instantaneous ellipse*.

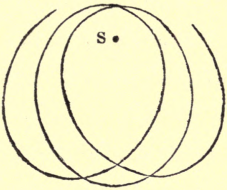
(44.) If the disturbing force ceases, the planet continues to revolve in the same ellipse, and the permanent ellipse coincides with the instantaneous ellipse corresponding to the instant when the disturbing force ceases.

(45.) If the disturbing force continues to act, the dimensions of the instantaneous ellipse are continually changing; but in the course of a single revolution, (even for our moon,) the dimensions alter so little, that the motion in the instantaneous ellipse corresponding to any instant during that revolution will very nearly agree with the real motion during that revolution.

We shall now consider the effects of particular forces in altering the elements.

(46.) (I.) Suppose that the disturbing force is always directed to the central body. The effect of this would be nearly the same as if the attraction or the mass of the central body was increased. The result of this on the dimensions of the orbit will be different according to the part of the orbit where it begins to act, and may be gathered from the cases to be mentioned separately hereafter, (we do not insist on it at present, as there is no instance in the planetary system of such sudden commencement of force.) But at all events the relation between the mean distance and the periodic time will not be the same as before; the time will be less for the same mean distance, or the mean distance greater for the same periodic time, than if the disturbing force did not act (38.). If the disturbing force is always directed from the central body, the effect will be exactly opposite. If the disturbing force does not alter, except with the planet's distance, the planet will at every successive revolution describe an orbit of the same size. For, as we have stated, (29.) the radius vector will in equal times pass over equal areas, and mathematicians have proved that, if the variation of force depends only on the distance, the velocity of the planet will depend only on the distance; and the consideration which determines the greatest or least distance of the planet is, that the planet, moving with the velocity which is proper to the distance, cannot describe the proper area in a short time, unless it move in the direction perpendicular to the radius vector. This consideration will evidently give the same values for the greatest and least distances at every revolution. It may happen that all the greatest distances will not be at the same place; the body may describe such an orbit as that in *fig.* 6.

Fig. 6.

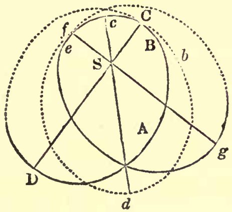


(47.) (II.) If, however, the disturbing force directed to the central body increases gradually and constantly during many revolutions, there is no difficulty in seeing that the planet will at every revolution be drawn nearer to the central body, and thus it will move, at every succeeding revolution, in a smaller orbit than at the preceding one; and will consequently perform its revolution in a shorter time. If the disturbing force directed to the central body diminishes, the orbit will become larger, and the periodic time longer. In the same manner, if the disturbing force is directed from the central body, a gradual increase of the disturbing force will increase the dimensions of the orbit and the periodic time, and a gradual diminution of the disturbing force will diminish the dimensions of the orbit and the periodic time.

(48.) (III.) Suppose that the disturbing force acts always in the direction in which the planet is distance has been altered in the proportion of $10000 : 10001$, the periodic time will have been altered in the proportion of $10000 : 10001\frac{1}{2}$ nearly, or the mean motion will have been altered in the proportion of $10001\frac{1}{2}$ to 10000 or $1 : 0{\cdot}99985$ nearly. If this alteration has gone on uniformly, we may suppose the whole motion in the 100 revolutions to have been nearly the same as if the planet had moved with a mean motion, whose value is half way between the values of the first and the last, or $0{\cdot}999925 \times$ the original mean motion. Therefore, at the time when we should expect the planet to have made 100 revolutions, it will only have made 99·9925 revolutions, or will be *behind* the place where we expected to see it by 0·0075 revolution, or nearly three degrees; a quantity which could not fail to be noticed by the coarsest observer. To use a borrowed illustration, the alteration of the mean distance in an orbit produces the same kind of effect as the alteration of the length of a clock pendulum: which, though so small as to be insensible to the eye, will, in a few days, produce a very great effect on the time shown by the clock.

(50.) (V.) Now suppose the orbit of the planet or satellite to be an ellipse; and suppose a disturbing force directed to the central body to act upon the planet, &c. only when it is near its perihelion or perigee, &c. In *fig.* 7, let A B be

Fig. 7.



the curve in which the planet is moving, and let the dotted line B C D A represent the orbit in which it would have moved if no disturbing force had acted, C being the place of perihelion. At B let the disturbing force, directed towards S, begin to act, and let it act for a little while and then cease. The planet is at that place approaching toward the sun, and the direction of its motion makes an acute angle with S B. It is evident that the disturbing force, which draws the planet more rapidly towards the sun without otherwise affecting its motion, will cause it to move in a direction that makes a more acute angle with S B. The part of the new path, therefore, which is nearest to the sun (that is, the new perihelion) will be farther from B than the perihelion C of the orbit in which the planet would have moved.

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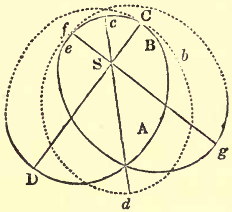


the curve in which the planet is moving, and let the dotted line B C D A represent the orbit in which it would have moved if no disturbing force had acted, C being the place of perihelion. At B let the disturbing force, directed towards S, begin to act, and let it act for a little while and then cease. The planet is at that place approaching toward the sun, and the direction of its motion makes an acute angle with S B. It is evident that the disturbing force, which draws the planet more rapidly towards the sun without otherwise affecting its motion, will cause it to move in a direction that makes a more acute angle with S B. The part of the new path, therefore, which is nearest to the sun (that is, the new perihelion) will be farther from B than the perihelion C of the orbit in which the planet would have moved.

The reader's conception of this will be facilitated by supposing the orbit instead of a curve to be a straight line, and the place of perihelion to be determined by letting fall a perpendicular from the sun upon the line; when it will be seen that, on drawing the line more acutely inclined to S B, the distance of the foot of the perpendicular from B is increased. With a curved orbit the result is just the same. In other words, the planet, instead of describing B C, will, in consequence of the action of the disturbing force, describe B *c*; and the place of perihelion, instead of C, will be *c*, a point more distant from B than C is. Now, if the disturbing force should not act again, the planet would move in an ellipse *cdb*, and the line of apses, instead of C S D, would be *c* S *d*. The line of apses has therefore twisted round in the same angular direction as that in which the planet was going; and this is expressed by saying that *the line of apses progresses*. If, after passing *c*, the disturbing force should again act for a little while, at *e* for instance, the recess of the planet from the sun would be diminished, its path would be more nearly perpendicular to the radius vector, and therefore the inclination of the path would be such as corresponds to a smaller distance from perihelion than the planet really has.

That is, when the planet leaves *e*, the inclination of its path to the radius vector is greater than it would have been if the planet had continued to move in the orbit *c d b*, but is the same as if its perihelion had been at some such situation as *f*, supposing no disturbing force to act. Now let the disturbing force cease entirely to act; and the planet, which at *e* is moving as if it had come from the perihelion *f*, will continue to move as if it had come from the perihelion *f*; it will proceed, therefore, to describe an elliptic orbit in which *f* S *g* is the line of apses: the line of apses has been twisted round in the same direction as before, or the line of apses has still progressed. The effect then of a disturbing force directed to the central body before and after passing the perihelion, is to make the line of apses progress *.

* This result, and those which follow immediately, may be inferred from the construction in Newton's 'Principia,' book i. sect. 3, prop. xvii. If we assume (as we suppose in all these investigations) the excentricity to be small, the disturbing force directed to the sun will not sensibly alter the planet's velocity, but will change the direction of its path at P, the place of action, (in Newton's figure ;) the length of P H, therefore, will not be altered, (since that length depends only on the velocity,) but its position will be altered, the position of P H being determined by making the angle R P H equal to the supplement of R P S. On trying the effects of this in different positions of P, and observing that the immediate effect of a disturbing force directed to the centre is to increase the rate of

(51.) In the same manner it will be seen, that the effect of a disturbing force, directed from the central body before and after passing the perihelion, is to make the line of apses regress.

(52.) The motion of the planet, subject to such forces as we have mentioned, would be *nearly* the same as if it was revolving in an elliptic orbit, and this elliptic orbit was at the same time revolving round its focus, turning in the same direction as that in which the planet goes round, and always carrying it on its circumference. And this is the easiest way of representing to the mind the *general effect* of this motion; the *physical cause* is to be sought in such explanations as that above.

(53.) (VI.) Suppose a disturbing force directed to the centre, to act upon the planet when it is near aphelion. As the planet is going towards aphelion it is receding from the sun. The effect of the disturbing force is to diminish the rate of recess from the sun; and, therefore, to increase the inclination of the planet's path to the radius vector. The aphelion is the place where

approach, or to diminish the rate of receding, and that the effect of a force directed from the centre is the opposite, all the cases in the text will be fully explained.

the planet's path is perpendicular to the radius vector. The effect of the disturbing force, then, which increases the inclination of the planet's path to the radius vector, will be to make that path perpendicular to the radius vector sooner than if the disturbing force had not acted. That is, the planet will be at aphelion sooner than it would have been if no disturbing force had acted. The aphelion has, as it were, gone backwards to meet the planet. If the disturbing force should entirely cease, the planet will move in an elliptic orbit, of which this new aphelion would be the permanent aphelion. The line passing through the aphelion has, therefore, twisted in a direction opposite to the planet's motion, or *the line of apses has regressed.* After passing aphelion, if the disturbing force still continues to act, the planet's approach to the sun will be quickened by the disturbing force, and, therefore, after some time, the planet's rate of approach will be greater than that corresponding, in an undisturbed orbit, to its actual distance from aphelion, and will be equal to that corresponding in an undisturbed orbit to a greater distance from aphelion. If, now, the disturbing force ceases, the planet, moving as if it came in an undisturbed orbit from an imaginary aphelion, will continue to move as if it came from that imaginary aphelion; and that imaginary aphelion having been at a greater distance behind the planet than the real aphelion, its place will be represented by saying that the line of apses has still regressed. The effect, then, of a disturbing force directed to the central body, before and after passing aphelion, is to make the line of apses regress.

(54.) In the same manner it will be seen, that the effect of a disturbing force, directed from the central body, before and after passing the aphelion, is to make the line of apses progress.

(55.) (VII.) Since a disturbing force, directed to the central body, or one directed from the central body, produces opposite effects with regard to the motion of the line of apses, according as it acts near perihelion or near aphelion, it is easy to perceive that there must be some place between perihelion and aphelion, where the disturbing force, directed to the central body, will produce no effect on the position of the line of apses. It is found by accurate investigation, that this point is the place where the radius vector is perpendicular to the line of apses *.

(56.) (VIII.) The effects mentioned above are greatest when the excentricity is small. Thus, if we compared two orbits, as figures 8 and 9, in one of which the excentricity was great, and in the other small; and if (for instance) we supposed the disturbing force to act for a short

Fig. 8.

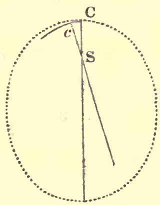
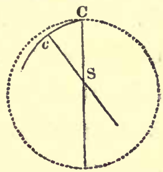


Fig. 9.



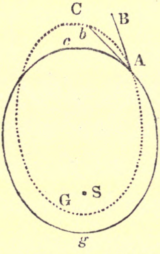
time at the perihelion C, and supposed the forces in the two orbits to be such as to deflect

* To the reader who is familiar with Newton's "Principia," sect. 3, the following demonstration will be sufficient:—The disturbing force, which is entirely in the direction of the radius vector, will not alter the area described in a given time, and, therefore, will not alter the *latus rectum* (to the square root of which the area is proportional.) But half the *latus rectum* of the undisturbed orbit is the radius vector at the supposed place of action of the disturbing force (since that radius vector is supposed perpendicular to the major axis.) Therefore, half the *latus rectum* of the new orbit is the radius vector at the point in question; and, consequently, the radius vector, at the point in question, is perpendicular to the major axis in the new orbit; but it was so in the undisturbed orbit; and, therefore, the major axes in the new orbit and the undisturbed orbit coincide.

the new paths from the old orbits by equal angles in the two cases; it is plain, that in *fig.* 8, in consequence of the curvature at C differing much from that of a circle whose centre is S, we should find the new perihelion *c* at a small distance from C; whereas in *fig.* 9, where the orbit does not differ much from the circle whose centre is S, *c* would be far removed from C. In fact, *c* would in both cases bisect the part of the orbit lying within that circle; and it is evident, that the angle at C being the same in both, the length of the part lying within the circle would be much less in *fig.* 8, where the orbit is almost a straight line, than in *fig.* 9, where the curvature of the orbit differs little from that of the circle. Or we may state it thus:—The alteration of the place of perihelion, or aphelion, depends on the proportion which the alteration in the approach or recess produced by the disturbing force bears to the whole approach or recess; and is, therefore, greatest when the whole approach or recess is least; that is, when the orbit is little excentric.

(57.) (IX.) To judge of the effect which a disturbing force, directed to the sun, will produce on the excentricity of a planet's orbit, let us suppose the planet to have left its perihelion, and to be moving towards aphelion, and, consequently, to be receding from the sun, and now let the disturbing force act for a short time. This will cause it to recede from the sun more slowly than it would have receded without the action of the disturbing force; and, consequently, the planet, without any material alteration in its velocity, (and, therefore, without any material alteration in the major axis of its orbit (28),) will be moving in a path more inclined to the radius vector than if the disturbing force had not acted. The planet may, therefore, be considered as projected from the point A, *fig.* 10., in the direction A *b* instead of A B, in which it was moving; and, therefore, instead

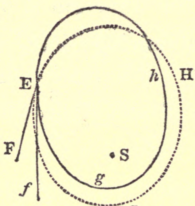
Fig. 10.



of describing the orbit A C G, in which it was moving before, it will describe an orbit A *c g*, more resembling a circle, or less excentric than before. The effect, therefore, of a disturbing force directed to the centre, while a planet is moving from perihelion to aphelion, is to diminish the excentricity of the orbit.

(58.) If we suppose the planet to be moving from aphelion to perihelion, it is approaching to the sun; the disturbing force directed to the sun makes it approach more rapidly; its path is, therefore, less inclined to the radius vector than it would have been without the disturbing force; and this effect may be represented by supposing that at E, *fig.* 11., instead of moving in the direction E F in which it was moving, the planet is projected in the direction E*f*. Instead, therefore, of describing the ellipse E G H, in which it was moving before, it will describe such an ellipse as E *g h*, which is more excentric than the former. The effect, therefore, of a disturbing force directed to the centre, while a planet is moving from aphelion

Fig. 11.



to perihelion, is to increase the excentricity of the orbit.

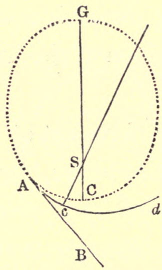
(59.) In a similar manner it will appear, that the effect of a disturbing force, directed from the centre, is to increase the excentricity as the planet is moving from perihelion to aphelion, and to diminish it as the planet moves from aphelion to perihelion.

(60.) (X.) Let us now lay aside the consideration of a force acting in the direction of the radius vector, and consider the effect of a force acting perpendicularly to the radius vector, in the direction in which the planet is moving. And first, its effect on the position of the line of apses.

(61.) If such a force act at one of the apses, either perihelion or aphelion, for a short time, it is clear that its effect will be represented by supposing that the velocity at that apse is suddenly increased, or that the velocity with which the planet is projected from perihelion is greater than the velocity with which it would have been projected if no disturbing force had acted. This will make no difference in the position of the line of apses; for with whatever velocity the planet is projected, if it is projected in a direction perpendicular to the radius vector, (which is implied in our supposition, that the place where the force acts was an apse in the old orbit,) the place of projection will infallibly be an apse in the new orbit; and the line of apses, which is the line drawn from that point through the centre, will be the same as before.

(62.) But if the force act for a short time before the planet reaches the perihelion, its principal * effect will be to increase its velocity; the sun's attraction will, therefore, have less power to curve its path (25.); the new orbit will be, in that part, exterior to the old one. In *fig.* 12., we must,

Fig. 12.



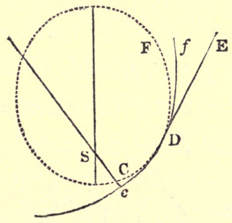
therefore, suppose that the planet, after leaving A, where the force has acted to accelerate its motion,

* It is supposed here, and in all our investigations, that the excentricity of the orbit is small, and, consequently, that a force perpendicular to the radius vector produces nearly the same effect as a force acting in the direction of a tangent to the ellipse.

instead of describing the orbit A C G, proceeds to describe the orbit A *c d*, which at A has the same direction (or has the same tangent A B) as the orbit A C G. It is plain now that *c* is the part nearest to the sun, or *c* is the perihelion: and it is evident here, that the line of apses has altered its position from S C to S *c*, or has twisted in a direction opposite to the angular motion of the planet, or has regressed.

(63.) If the force act for a short time after the planet has passed perihelion, as at D in *fig.* 13, the planet's velocity is increased there, and the path described by the planet is D*f*, instead of

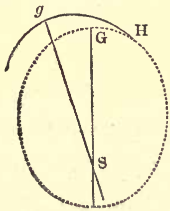
Fig. 13.



D F, having the same direction at D, (or having the same tangent D E,) but less curved, and, therefore, exterior to D F. If now we conceive the planet to have received the actual velocity with which it is moving in D *f*, from moving without disturbance in an elliptic orbit *c* D*f*, (which is the orbit that it will now proceed to describe, if no disturbing force continues to act,) it is evident that the part *c* D must be described with a greater velocity than C D, inasmuch as the velocity at D from moving in *c* D is greater than the velocity from moving in C D; *c* D is, therefore, less curved than C D, and, therefore, exterior to it, (since it has the same direction at D;) and then the perihelion is some point in the position of *c*, and the line of apses has changed its direction from S C to S *c*, or has twisted round in the same direction in which the planet is moving, or has progressed.

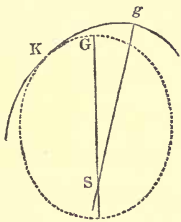
(64.) If the force act for a short time before passing aphelion, it will be seen in the same manner that the line of apses is made to progress. It is only necessary to consider that (as before) the new orbit has the same direction at the point H, *fig.* 14, where the force has acted as the old one,

Fig. 14.



but is less curved, and, therefore, exterior to it; and the aphelion, or point most distant from the sun, is *g* instead of G, and the position of the line of apses has shifted from S G to S *g*. If the force act after the planet has passed aphelion, as at K, *fig.* 15, the orbit in which we must conceive the planet to have come, in order to have the increased velocity, must be *g* K exterior to G K; the point

Fig. 15.



most distant from the sun must be *g* instead of G, and the line of apses must have changed from S G to S *g*, or must have regressed.

(65.) Collecting these conclusions *, we see that, if a disturbing force act perpendicularly to the radius vector, in the direction in which the planet is moving, its action, while the planet passes from perihelion to aphelion, causes the line of apses to progress; and its action, while the planet passes

* These conclusions, and those that follow, will be easily inferred from Newton's construction, Prop. XVII., by observing, that an increase of the velocity increases the length of P H in Newton's figure without altering its position.

from aphelion to perihelion, causes the apses to regress.

(66.) By similar reasoning, if the direction of the disturbing force is opposite to that in which the planet is moving, its action, while the planet passes from perihelion to aphelion, causes the line of apses to regress, and while the planet passes from aphelion to perihelion causes the apses to progress.

(67.) (XI.) For the effect on the excentricity: suppose the disturbing force, increasing the velocity, to act for a short time at perihelion; the effect is the same as if the planet were projected from perihelion with a greater velocity than that which would cause it to describe the old orbit. The sun's attraction, therefore, will not be able to pull it in into so small a compass as before; and at the opposite part of its orbit, that is, at aphelion, it will go off to a greater distance than before; but as it is moving without disturbance, and, therefore, in an ellipse, it will return to the same perihelion. The perihelion distance, therefore, remaining the same, and the aphelion distance being increased, the inequality of these distances is increased, and the orbit, therefore, is made more excentric. Now, suppose the force increasing the velocity to act at aphelion. Just as before, the sun's attraction will be unable to make the planet describe an orbit so small as its old orbit, and the distance at the opposite point (that is, at perihelion) will be increased; but the planet will return to the same aphelion distance as before. Here, then, the inequality of distances is diminished, and the excentricity is diminished.

(68.) Thus we see that a disturbing force, acting perpendicularly to the radius vector, in the direction of the planet's motion, increases the excentricity if it acts on the planet near perihelion, and diminishes the excentricity if it acts on the planet near aphelion. And, similarly, if the force acts in the direction opposite to that of the planet's motion, it diminishes the excentricity by acting near perihelion, and increases it by acting near aphelion.

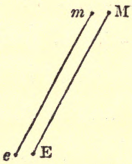
(69.) (XII.) In all these investigations, it is supposed that the disturbing force acts for a very short time, and then ceases. In future, we shall have to consider the effect of forces, which act for a long time, changing in intensity, but not ceasing. To estimate their effect we must suppose the long time divided into a great number of short times; we must then infer, from the preceding theorems, how the elements of the *instantaneous ellipse* (43.) are changed in each of these short times by the action of the force, which is then disturbing the motion; and we must then recollect, that the instantaneous ellipse, at the end of the long time under consideration, will be the same as the permanent ellipse in which the planet will move, if the disturbing force then ceases to act (43.), and that it will, at all events, differ very little from the curve described in the next revolution of the planet, even if the disturbing force continue to act (41.)

Section IV.—*On the Nature of the Force disturbing a Planet or Satellite, produced by the Attraction of other Bodies.*

(70.) Having examined the effects of disturbing forces upon the elements of a planet's or satellite's orbit, we have now to inquire into the kind of the disturbing force which the attraction of another body produces. The inquiry is much simpler than might at first sight be expected; and this simplicity arises, in part, from the circumstance that (as we have mentioned in (6.)) the attraction of a planet upon the sun is the same as its attraction upon another planet, when the sun and the attracted planet are equally distant from the attracting planet.

(71.) First, then, we have to remark, that the disturbing force is not the whole attraction. The sun, for instance, attracts the moon, and disturbs its elliptic motion round the earth; yet the force which disturbs the moon's motion is not the whole attraction of the sun upon the moon. For the effect of the attraction is to move the moon from the place where it would otherwise have been; but the sun's attraction upon the earth also moves the earth from the place where it would otherwise have been; and if the alteration of the earth's place is exactly the same as the alteration of the moon's place, the relative situation of the earth and moon will be the same as before. Thus, if, in *fig.* 16, any attraction carries the earth from E to *e*, and carries the moon from M to *m*, and if E *e* is equal and parallel to M *m*, then *e m*, which is the distance of the earth and moon, on the supposition that the attraction acts on both, is equal to E M,

Fig. 16.



which is their distance, on the supposition that the attraction acts on neither; and the line *e m*, which represents the direction in which the moon is seen from the earth, if the attraction acts on both, is parallel to E M, which represents the direction in which the moon is seen from the earth, if the attraction acts on neither. The distance, therefore, of the earth and moon, and the direction in which the moon is seen from the earth, being unaltered by such a force, their relative situation is unaltered. An attraction, therefore, which acts equally, and in the same direction, on both bodies, does not disturb their relative motions.

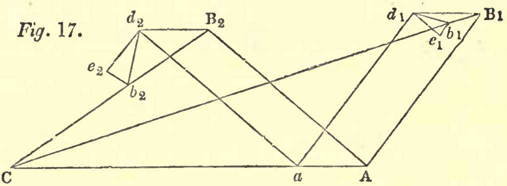
From this we draw the two following important conclusions:—

(72.) Firstly. A planet may revolve round the sun, carrying with it a satellite, and the satellite may revolve round the planet in nearly the same manner as if the planet was at rest. For the attraction of the sun on the planet is nearly the same as the attraction of the sun on the satellite. It is true that they are not exactly the same, and the effects of the difference will soon form an important subject of inquiry; but they are, upon the whole, very nearly the same. The moon is sometimes nearer to the sun than the earth is, and sometimes farther from the sun; and, therefore, the sun's attraction on the moon is sometimes greater than its attraction on the earth, and sometimes less; but, upon the whole, the inequality of attractions is very small. It is owing to this that we may consider a satellite as revolving round a planet in very nearly the same manner (in respect of relative motion) as if there existed no such body as the sun.

(73.) Secondly. The force which disturbs the motion of a satellite, or a planet, is the difference of the forces (measured, as in (4.), by the spaces through which the forces draw the bodies respectively) which act on the central and the revolving body. Thus, if the moon is between the sun and the earth, and if the sun's attraction in a certain time draws the earth 200 inches, and in the same time draws the moon 201 inches, then the real disturbing force is the force which would produce in the moon a motion of one inch from the earth.

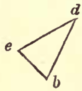
(74.) In illustrating the second remark, we have taken the simplest case that can well be imagined. If, however, the moon is in any other situation with respect to the earth, some complication is introduced. Not only is the moon's distance from the sun different from the earth's distance, (which according to (9.) produces an inequality in the attractions upon the earth and moon,) but also the direction in which the attraction acts on the earth is different from the direction in which it acts on the moon, (inasmuch as the attraction always acts in the direction of the line drawn from the attracted body to the attracting body; and the lines so drawn from the earth and moon to the sun are in different directions.) The same applies in every respect to the perturbation which one planet produces in the motion of a second planet round the sun, and which depends upon the difference in the first planet's attractions upon the sun and upon the second planet. To overcome this difficulty we must have recourse to geometrical considerations. In *fig.* 17., let B_1 be a body revolving about A, and

Fig. 17.



let C be another body whose attraction disturbs the motion of B_1 round A. The attraction of C will in a certain time draw A to a; it will in the same time draw B_1 to b_1. Make $B_1 d_1$ equal and parallel to A a; then $a d_1$ will be equal and parallel to A B_1. Now if the force upon B_1 were such as to draw it to d_1, the motion of B_1 round A would not be disturbed by that force. But the force upon B_1 is really such as to draw it to b_1. The real disturbing force then may be represented as a force which draws the revolving body from d_1 to b_1. If, instead of supposing the revolving body to be at B_1 we suppose it at B_2, and if the attraction of C would draw it through $B_2 b_2$ while it draws A through A a, then (in the same manner, making $B_2 d_2$ equal and parallel to A a) the real disturbing force may be represented by a force which in the same time would draw B_2 through $d_2 b_2$.

(75.) Both the magnitude and the direction of this force are continually varying, and we must, if possible, find a convenient way of representing it. We shall have recourse here to the "composition

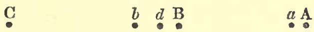
Fig. 18. 

of motion." In *fig.* 18., if *d b* represent the space through which a force has drawn a body in a certain time, the same effect may be produced by two forces of which one would in the same time draw the body from *d* to *e*, and the other would in the same time draw the body from *e* to *b*. And this is true whatever be the directions and lengths of *d e* and *e b*, provided that with *d b* they form a triangle. To accommodate the investigations of this Section to those of Section III., we will suppose *d e* perpendicular to the radius vector, and *eb* parallel to the radius vector. In *fig.* 17. draw *d e* perpendicular to A B or *a d*, and *e b* parallel to A B or *a d;* and now we can say: the disturbing force produced by the attraction of C is a force represented by *d e* perpendicular to the radius vector, and a force represented by *e b* in the direction of the radius vector.

(76.) We now want nothing but estimations of the magnitudes of these forces in order to apply the investigations of Section III. For the present we shall content ourselves with pointing out some of the most interesting cases.

(77.) I. Let the disturbing body be exterior to the orbit of the disturbed body: (this applies to the disturbance of the moon's motion produced by the sun's attraction, the disturbance of the earth's motion by Jupiter's attraction, the disturbance of the motion of Venus by the earth's

Fig. 19.



attraction, &c.:) and first, let the revolving body B be between the disturbing body C and the central body A (as in *fig.* 19.) If the attraction of C will in a certain time draw A to *a*, it will in the same time draw B to *b*, where B *b* is much greater than A *a*. Take B *d* equal to A *a*, then *d b* is the effect of the disturbing force, which tends to draw B further from A. In this case then, the disturbing force is entirely in the direction of the radius vector, and directed *from* the central body. This is the greatest disturbing force that can be produced by C.

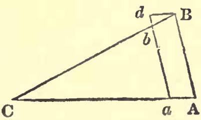
(78.) II. Let C A B (*fig.* 20.) be in the same

Fig. 20.

straight line, but let B be on the side of A, opposite to C. In this case B *b* is less than A *a*; and if B *d* is taken equal to A *a*, the disturbing force represented by *d b* will be entirely in the direction of the radius vector, and directed from the central body. This case is particularly deserving of the reader's consideration, as the *effectual disturbing* force is exactly opposite to the attraction which C actually exerts upon B.

(79.) III. The disturbing force in the case represented in *fig.* 19. is much greater than that in the case of *fig.* 20., except C be very distant. Thus, suppose A B to be half of A C. In the first case, the attraction upon B (by the law of gravitation) is four times as great as the attraction upon A, and therefore the disturbing force (which is the difference of the forces on A and B) is three times as great as the attraction upon A. In the second case, the distance of B is $\frac{3}{2}$ of the distance of A, and therefore the attraction upon B is $\frac{4}{9}$ of the attraction upon A, and the disturbing force is $\frac{5}{9}$ of the attraction upon A. The disturbing force in the first case is, therefore, greater than in the second case, in the proportion of 3 to $\frac{5}{9}$, or 27 to 5. This remark applies to nearly all the cases of planetary disturbance where the disturbing planet is exterior to the orbit of the disturbed planet, the ratio between these distances from the sun being a ratio of not very great inequality. But it scarcely applies to the moon. For the sun's distance from the earth is nearly 400 times the moon's distance: consequently when the moon is between the sun and the earth, the attraction of the sun on the moon is $(\frac{400}{399})^2 \times$ the attraction of the sun on the earth, or $\frac{160000}{159201}$ parts of the sun's attraction on the earth, and the disturbing force therefore is $\frac{799}{159201}$ parts of the sun's attraction on the earth: but when the moon is on that side farthest from the sun, the sun's attraction on the moon is $(\frac{400}{401})^2$ or $\frac{160000}{160801}$ parts of the sun's attraction on the earth, and the disturbing force is $\frac{801}{160801}$ parts of the sun's attraction on the earth, which is very little less than the former. The effects of the difference are, however, sensible.

Fig. 21.



21 (80.) IV. Suppose B, *fig.* 26, to be in that part of its orbit which is at the same distance from C as the distance of A from C. The attraction of C upon the two other bodies, whose distances are equal, will be equal, but not in the same direction. B *b*, therefore, will be equal to A *a*. But since C B is also equal to C A, it is evident that *a b* will be parallel to A B, and therefore *b* will be in the line *a d*. Consequently in this case also the disturbing force will be entirely in the direction of the radius vector: but here, unlike the other cases, the disturbing force is directed *towards* the central body. The magnitude of the disturbing force bears the same proportion to the whole attraction on A which *b d* bears to B *b*, or A B to A C. Thus, in the first numerical instance taken above, the disturbing force in this part of the orbit is $\frac{1}{2}$ of the attraction on A: and in the second numerical instance, the disturbing force is $\frac{1}{400}$ of the attraction on A. It is important to observe that in both instances the disturbing force, when wholly directed *to* the centre, is much less than either value of the disturbing force when wholly directed *from* the centre: in the latter instance it is almost exactly one-half.

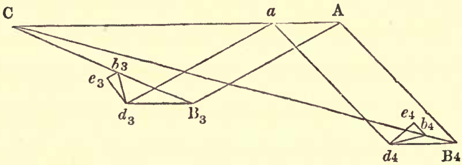
(81.) When the disturbing body is distant, the point of the orbit which we have here considered is very nearly that determined by drawing A B perpendicular to C A.

(82.) V. When C is distant, (as in the case of the moon disturbed by the sun,) the disturbing forces mentioned in (III.) and (IV.) are nearly proportional to the distance of the moon from the earth. For the force mentioned in (IV.) this is exactly true, whether C be near or distant, because (as we have found) the disturbing force bears the same proportion to the whole attraction on A which A B bears to A C. With regard to the force mentioned in (III.); if we suppose the moon's distance from the earth to be $\frac{1}{400}$ of the sun's distance, the disturbing force when the moon is between the earth and the sun is $\frac{799}{159201}$ parts of the sun's attraction on the earth, or nearly $\frac{1}{200}$th part. But if we suppose the moon's distance from the earth to be $\frac{1}{200}$th of the sun's distance, the attraction on the moon (when between the earth and the sun) would be $(\frac{200}{199})^2$ or $\frac{40000}{39601}$ parts of the attraction on the earth; the disturbing force, or the difference of attractions on the earth and moon, would be $\frac{399}{39601}$, or nearly $\frac{1}{100}$th part of the sun's attraction on the earth. Thus, on doubling the moon's distance from the earth, the disturbing force is nearly doubled: and in the same manner, on altering the distance in any other proportion, we should find that the disturbing force is altered in nearly the same proportion.

(83.) VI. If, while the moon's distance from the earth is not sensibly altered, the earth's distance from the sun is altered, the disturbing force is diminished very nearly in the same ratio in which the cube of the sun's distance is increased. For if the sun's distance is 400 times the moon's distance, and the moon between the earth and the sun, we have seen that the disturbing force is nearly $\frac{1}{200}$th part of the sun's attraction on the earth *at that distance of the sun.* Now, suppose the sun's distance from the earth to be made 800 times the moon's distance, or twice the former distance: the sun's distance from the moon will be 799 times the moon's distance, or $\frac{799}{400}$ parts of the sun's former distance from the earth; the attractions on the earth and moon respectively will be $\frac{1}{4}$ and $\frac{160000}{638401}$ parts of the former attraction on the earth: and the disturbing force, or the difference between these, will be $\frac{1599}{2553604}$, or nearly $\frac{1}{1600}$th part of the former attraction of the earth. Thus, on doubling the sun's distance, the disturbing force is diminished to $\frac{1}{8}$th part of its former value; and a similar proposition would be found to be true if the sun's distance were altered in any other proportion.

(84.) VII. Suppose B to have moved from that part of its orbit where its distance from C is equal to A's distance from C, towards the part where it is between A and C. Since at the point where B's distance from C is equal to A's distance from C, the disturbing force is in the direction of the radius vector, and directed *towards* A, and since at the point where B is between A and C, the disturbing force is in the direction of the radius vector, but directed *from* A, it is plain that there is some situation of B, between these two points, in which there is no disturbing force at all in the direction of the radius vector. On this we shall not at present speak further: but we shall remark that there is a disturbing force perpendicular to the radius vector, at every such intermediate point. This will be easily seen from the second case of *fig.* 17. On going through the reasoning in that place it will appear that, between the two points that we have mentioned, there is always a disturbing force $d_2\,e_2$ perpendicular to the radius vector, and in the same direction in which the body is going. If now we construct a similar figure for the situation B_3, *fig.* 22., in which B is moving from the

Fig. 22.

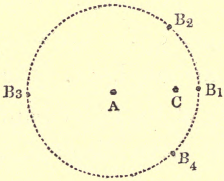


point between C and A to the other point whose distance from C is equal to A's distance from C, we shall find that there is a disturbing force $d_3\,e_3$ perpendicular to the radius vector, in the direction opposite to that in which B is going. If we construct a figure for the situation B_4 in which B is moving from the point of equal distances, to the point where B is on the side of A opposite to C, we shall see that there is a disturbing force perpendicular to the radius vector, in the same direction in which B is going; and in the same manner, for the situation B_1 in *fig.* 17. where B is moving from the point on the side of A opposite C to the next point of equal distances, there is a disturbing force perpendicular to the radius vector, in the direction opposite to that in which B is going.

(85.) The results of all these cases may be collected thus. The disturbing body being exterior to the orbit of the revolving body, there is a disturbing force in the direction of the radius vector only, directed *from* the central body, at the points where the revolving body is on the same side of the central body as the disturbing body, or on the opposite side, (the force in the former case being the greater,) and directed *to* the central body, at each of the places where the distance from the disturbing body is equal to the distance of the central body from the disturbing body. The force directed to the central body at the latter points, is however much less than the force directed from it at the former. Between the adjacent pairs of these four points there are four other points, at which the disturbing force in the direction of the radius vector is nothing. But while the revolving body is moving from one of the points, where it is on the same side of the central body as the disturbing body, or on the opposite side, to one of the equidistant points, there is always a disturbing force perpendicular to the radius vector tending to retard it; and while it is moving from one of the equi-distant points to one of the points on the same side of the central body as the disturbing body, or the opposite, there is a disturbing force perpendicular to the radius vector tending to accelerate it.

(86.) VIII. Now, let the disturbing body be supposed interior to the orbit of the revolving body, (as, for instance, when Venus disturbs the motion of the earth.) If B is in the situation B_1, *fig.* 23, the attraction of C draws A strongly towards B_1, and B_1 strongly towards A, and,

Fig. 23.



therefore, there is a very powerful disturbing force drawing B_1 towards A. If B is in the situation B_3, the attraction of C draws A strongly from B_3, and draws B_3 feebly towards A; therefore, there is a small disturbing force drawing B_3 from A. At some intermediate points the disturbing force in the direction of the radius vector is nothing. With regard to the disturbing force perpendicular to the radius vector: if A C is greater than $\frac{1}{2}$ A B_1, it will be possible to find two points, B_2 and B_4, whose distance from C is equal to the distance of A from C, and there the disturbing force perpendicular to the radius vector is nothing (or the whole disturbing force is in the direction of the radius vector). While B moves from the position B_1 to B_2, it will be seen by such reasoning as that of (75.) and (84.), that the disturbing force, perpendicular to the radius vector, retards B's motion; while B moves from B_2 to B_3, it accelerates B's motion; while B moves from B_3 to B_4 it retards B's motion; and while B moves from B_4 to B_1, it accelerates B's motion. But if A C is less than $\frac{1}{2}$ A B_1, there are no such points, B_2 B_4, as we have spoken of; and the disturbing force, perpendicular to the radius vector, accelerates B as it moves from B_1 to B_2, and retards B as it moves from B_2 to B_1.

We shall now proceed to apply these general principles to particular cases.

Section V.—*Lunar Theory.*

(87.) The distinguishing feature in the Lunar Theory is the general simplicity occasioned by the great distance of the disturbing body (the sun alone producing any sensible disturbance), in proportion to the moon's distance from the earth. The magnitude of the disturbing body renders these disturbances very much more conspicuous than any others in the solar system; and, on this account, as well as for the accuracy with which they can be observed, these disturbances have, since the invention of the Theory of Gravitation, been considered the best tests of the truth of the theory.

Some of the disturbances are independent of the excentricity of the moon's orbit; others depend, in a very remarkable manner, upon the excentricity. We shall commence with the former.

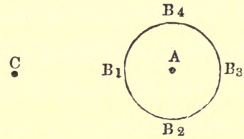
(88.) The general nature of the disturbing force on the moon may be thus stated. (See (77.) to (86.)) When the moon is either at the point between the earth and sun, or at that opposite to the sun (both which points are called syzygies), the force is entirely in the direction of the radius vector, and directed from the earth. When the moon is (very nearly) in the situations at which the radius vector is perpendicular to the line joining the earth and sun (both which points are called quadratures), the force is entirely in the direction of the radius vector, and directed to the earth. At certain intermediate points there is no disturbing force in the direction of the radius vector. Except at syzygies and quadratures, there is always a force perpendicular to the radius vector, such as to retard the moon while she goes from syzygy to quadrature, and to accelerate her while she goes from quadrature to syzygy.

(89.) I. As the disturbing force, in the direction of the radius vector, directed from the earth, is greater than that directed to the earth, we may consider that, upon the whole, the effect of the disturbing force is to diminish the earth's attraction. Thus the moon's mean distance from the earth is less (see (46.)) than it would have been with the same periodic time, if the sun had not disturbed it. The force perpendicular to the radius vector sometimes accelerates the moon, and sometimes retards it, and, therefore, produces no permanent effect.

(90.) II. But the sun's distance from the earth is subject to alteration, because the earth revolves in an elliptic orbit round the sun. Now, we have seen (83.) that the magnitude of the disturbing force is inversely proportional to the cube of the sun's distance; and, consequently, it is sensibly greater when the earth is at perihelion than when at aphelion. Therefore, while the earth moves from perihelion to aphelion, the disturbing force is continually diminishing; and while it moves from aphelion to perihelion, the disturbing force is constantly increasing. Referring then to (47.) it will be seen, that in the former of these times the moon's orbit is gradually diminishing, and that in the latter it is gradually enlarging. And though this alteration is not great (the whole variation of dimensions, from greatest to least, being less than $\frac{1}{5000}$), the effect on the angular motion (see (49.)) is very considerable; the angular velocity becoming quicker in the former time and slower in the latter; so that while the earth moves from perihelion to aphelion, the moon's angular motion is constantly becoming quicker, and while the earth moves from aphelion to perihelion the moon's angular motion is constantly becoming slower. Now, if the moon's mean motion is determined by comparing two places observed at the interval of many years, the angular motion so found is a mean between the greatest and least. Therefore, when the earth is at perihelion, the moon's angular motion is slower than its mean motion; and when the earth is at aphelion, the moon's angular motion is quicker than its mean motion. Consequently, while the earth is going from perihelion to aphelion, the moon's true place is always behind its mean place (as during the first half of that period the moon's true place is dropping behind the mean place, and during the latter half is gaining again the quantity which it had dropped behind); and while the earth is going from aphelion to perihelion, the moon's true place is always before its mean place. This inequality is called the moon's *annual equation;* it was discovered by Tycho Brahe from observation, about A.D. 1590; and its greatest value is about 10′, by which the true place is sometimes before and sometimes behind the mean place.

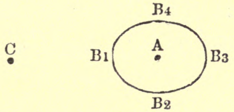
(91.) III. The disturbances which are periodical in every revolution of the moon, and are independent of excentricity, may thus be investigated. Suppose the sun to stand still for a few revolutions of the moon (or rather suppose the earth to be stationary,) and let us inquire in what kind of orbit, symmetrical on opposite sides, the sun can move. It cannot move in a circle: for the force perpendicular to the radius vector retards the moon as it goes from B_1 to B_2, *fig.* 24, and its velocity is, therefore, less at B_2 than at B_1, and on this account (sup-

Fig. 24.



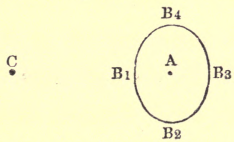
posing the force directed to A at B_2 equal to the force directed to A at B_1,) the orbit would be more curved at B_2 than at B_1. But the force directed to A at B_2 is much greater than that at B_1 (see (88.)); and on this account the orbit would be still more curved at B_2 than at B_1; whereas, in a circle, the curvature is every where the same. The orbit cannot, therefore, be circular. Neither can it be an oval with the earth in its centre, and with its longer axis passing through the sun, as *fig.* 25; for the velocity being small at B_2 (in consequence of the disturbing force perpendicular to the radius vector having retarded it,) while the earth's attraction is great (in consequence of the nearness of

Fig. 25.



B_2), and increased by the disturbing force in the radius vector directed towards the earth, the curvature at B_2 ought to be much greater than at B_1, where the velocity is great, the moon far off, and the disturbing force directed from the earth. But, on the contrary, the curvature at B_2 is much less than at B_1; therefore, this form of orbit is not the true one. But if the orbit be supposed to be oval, with its shorter axis directed towards the sun, as in *fig.* 26, all the conditions will be satisfied. For the velocity at B_2 is diminished by the disturbing force having

Fig. 26.



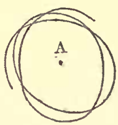
acted perpendicularly to the radius vector, while the moon goes from B_1 to B_2; and though the distance from A being greater, the earth's attraction at B_2 will be less than the attraction at B_1; yet, when increased by the disturbing

E 5

force, directed to A at B_2, it will be very little less than the attraction diminished by the disturbing force at B_1. The diminution of velocity then at B_2 being considerable, and the diminution of force small, the curvature will be increased; and this increase of curvature, by proper choice of the proportions of the oval, may be precisely such as corresponds to the real difference of curvature in the different parts of the oval. Hence, such an oval may be described by the moon without alteration in successive revolutions.

(92.) We have here supposed the earth to be stationary with respect to the sun. If, however, we take the true case of the earth moving round the sun, or the sun appearing to move round the earth, we have only to suppose that the oval twists round after the sun, and the same reasoning applies. The curve described by the moon is then such as is represented in *fig.* 27. As the disturb-

Fig. 27.



ing force, perpendicular to the radius vector, acts in the same direction for a longer time than in the former case, the difference in the velocity at syzygies and at quadratures is greater than in the former case, and this will require the oval to differ from a circle, rather more than if the sun be supposed to stand still.

(93.) If, now, in such an orbit as we have mentioned, the law of *uniform description of areas by the radius vector* were followed, as it would be if there were no force perpendicular to the radius vector, the angular motion of the moon near B_2 and B_4, *fig.* 26, would be much less than that near B_1 and B_2. But in consequence of the disturbing force, perpendicular to the radius vector, (which retards the moon from B_1 to B_2, and from B_3 to B_4, and accelerates it from B_2 to B_3, and from B_4 to B_1,) the angular motion is still less at B_2 and B_4, and still greater at B_1 and B_3. The angular motion, therefore, diminishes considerably while the moon moves from B_1 to B_2, and increases considerably while it moves from B_2 to B_3, &c. The mean angular motion, determined by observation, is less than the former and greater than the latter. Consequently, the angular motion at B_1 is greater than the mean, and that at B_2 is less than the mean; and, therefore, (as in (90.),) from B_1 to B_2 the moon's true place is before the mean; from B_2 to B_3 the true place is behind the mean; from B_3 to B_4 the true place is before the mean; and from B_4 to B_1 the true place is behind the mean. This inequality is called the moon's *variation*; it amounts to about 32′, by which the moon's true place is sometimes before and sometimes behind the mean place. It was discovered by Tycho, from observation, about A.D. 1590.

(94.) We have, however, mentioned, in (79.), that the disturbing forces are not exactly equal on the side of the orbit which is next the sun, and on that which is farthest from the sun; the former being rather greater. To take account of the effects of this difference, let us suppose, that in the investigation just finished, we use a mean value of the disturbing force. Then we must, to represent the real case, suppose the disturbing force near conjunction to be increased, and that near opposition to be diminished. Observing what the nature of these forces is, (77.), (78.), and (84.), this amounts to supposing that near conjunction the force necessary to make up the difference is a force acting in the radius vector, and directed from the earth, and a force perpendicular to the radius vector, accelerating the moon before conjunction, and retarding her after it, and that near opposition the forces are exactly of the contrary kind. Let us then lay aside the consideration of all other disturbing forces, and consider the inequality which these forces alone will produce. As they are very small, they will not in one revolution alter the orbit sensibly from an elliptic form. What then must be the excentricity, and what the position of the line of apses that, with these disturbing forces only, the same kind of orbit may always be described? A very little consideration of (57.), (58.), and (68.), will show, that unless the line of apses pass through the sun, the excentricity will either be increasing or diminishing from the action of these forces. We must assume, therefore, as our orbit is to have the same excentricity at each revolution, that the line of apses passes through the sun. But is the perigee or the apogee to be turned towards the sun? To answer this question we have only to observe, that the line of apses must progress as fast as the sun appears to progress, and we must, therefore, choose that position in which the forces will cause progression of the line of apses. If the perigee be directed to the sun, then the forces at both parts of the orbit will, by (51.), (54.), (65.), and (66.), cause the line of apses to regress. This supposition, then, cannot be admitted. But if the apogee be directed to the sun, the forces at both parts of the orbit will cause it to progress; and by (56.), if a proper value is given to the excentricity it will progress exactly as fast as the sun appears to progress. The effect, then, of the difference of forces, of which we have spoken, is to elongate the orbit towards the sun, and to compress it on the opposite side. This irregularity is called the *parallactic inequality.*

We shall shortly show, that if the moon revolved in such an elliptic orbit as we have mentioned, the effect of the other disturbing forces (independent of that discussed here) would be to make its line of apses progress with a considerable velocity. The force considered here, therefore, will merely have to cause a progression which, added to that just mentioned, will equal the sun's apparent motion round the earth. The excentricity of the ellipse, in which it could produce this smaller motion, will (56.) be greater than that of the ellipse in which the same force could produce the whole motion. Thus the magnitude of the parallactic inequality is considerably increased by the indirect effect of the other disturbing forces.

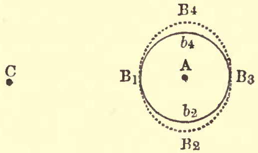
(95.) The magnitude of the forces concerned here is about $\frac{1}{130}$th of those concerned in (91.), &c.; but the effect is about $\frac{1}{15}$th of their effect. This is a striking instance of the difference of proportions in forces, and the effects that they produce, depending on the difference in their modes of action. The inequality here discussed is a very interesting one, from the circumstance that it enables us to determine with considerable accuracy the proportion of the sun's distance to the moon's distance, which none of the others will do, as it is found upon calculation, that their magnitude depends upon nothing but the excentricities and the proportion of the periodic times, all which are known without knowing the proportion of distances.

(96.) The effect of this, it will be readily understood, is to be combined with that already found. *See the note to* (134.) The moon's orbit, therefore, is more flattened on the side farthest from the sun, and less flattened on the side next the sun, than we found in (91.) and (92.) The equable description of areas is scarcely affected by these forces. The moon's *variation*, therefore, is somewhat diminished near conjunction, and is somewhat increased near opposition.

(97.) It will easily be imagined, that if there is an excentricity in the moon's orbit, the effect of the *variation* upon that orbit will be almost exactly the same as if there were no excentricity*.

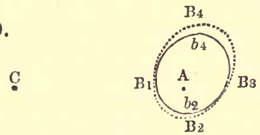
* As this general proportion is of considerable importance, we shall point out the nature of the reasoning by which (with proper alteration for different cases,) the reader may satisfy himself of its correctness. The reason why, in *fig.* 29, the moon cannot describe the circle B_1, b_2, B_3, b_4, though it touches at B_1 and B_3, and the reason that it will describe the oval B_1, B_2, B_3, B_4, is,

Fig. 29.



that the disturbing force makes the forces at B_1 and B_3 less than they would otherwise have been, and greater at B_2 and B_4 than they would otherwise have been; and the velocity is, by that part of the force perpendicular to the radius vector, made less at B_2, than it would otherwise have been. So that, unless we supposed it moving at B_1 with a greater velocity than it would have had, undisturbed, in the circle B_1, b_2, B_3, b_4, the great curvature produced by the great force, and diminished velocity at B_2, would have brought it much nearer to A than the point B_3; but with this large velocity at B_1, it will go out farther at B_2, and then the great curvature may make it pass exactly through B_3. In like manner, in *fig.* 30, if the velocity at B_1 were not

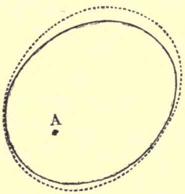
Fig. 30.



greater than it would have had, undisturbed, in the ellipse B_1, b_2, B_3, b_4, the increased curvature at B_2, produced by the

Thus, supposing that the orbit without the disturbing force had such a form as the dark line in *fig.* 28, it will, with the disturbing force, have such

Fig. 28.



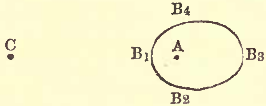
a form as the dotted line in that figure. The same must be understood in many other cases of different inequalities which affect the motion of the same body.

(98.) IV. We now proceed with the disturbances dependent on the excentricity: and, first, with the motion of the moon's perigee. In the first place, suppose that the perigee is on the same side as the sun. While the moon is near B_1, *fig.* 31, that is near perigee, the disturbing force

increased force and diminished velocity there, would have brought it much near to A than the point B_3; but with a large velocity at B_1 it will go out at B_2 further than it would otherwise have gone out, and then the increased force and diminished velocity will curve its course so much, that it may touch the elliptic orbit at B_3; and so on. The whole explanation, in one case as much as in the other, depends entirely upon the difference of the forces in the actual case, from the forces, if the moon were not disturbed.

is directed from A; and, consequently, by (51.), the line of apses regresses. While the moon is

Fig. 31.



near B_3, that is near apogee, the disturbing force is also directed from A, and, consequently, by (54.), the line of apses progresses. The question, then, now is, which is the greater, the regress, when the moon is near B_1, or the progress, when it is near B_3? To answer this, we will remark, that if the disturbing force directed from A, were inversely proportional to the square of the distance (and, consequently, less at B_3 than at B_1,) it would amount to exactly the same as if the attraction of A were altered in a given proportion*; and in that case B would

* The reasoning in the text may be more fully stated thus: If with the original attractive force of the earth there be combined another force, directed from the earth, and always bearing the same proportion to the earth's original attraction, this combined force may be considered in two ways: 1st, As a smaller attraction, always proportional to the original attraction, or inversely proportional to the square of the distance. 2d, As the original attraction, with a force superadded, which may be treated as a disturbing force. The result of the first mode of consideration will be, that the moon will describe an ellipse, whose line of apses does not move. The result of the second mode of consideration will be,

describe round A an ellipse, whose line of apses was invariable; or the progression produced at B_3 would be equal to the regression produced at B_1. But, in fact, the disturbing force at B_3 is to that at B_1 in the same proportion as $A B_3$ to $A B_1$, by (82.), and, therefore, the disturbing force at B_3 is greater than that at B_1; and, consequently, much greater than that which would produce a progression equal to the regression produced at B_1; and, therefore, the effects of the disturbing force at B_3 predominate, and the line of apses progresses. The

that the instantaneous ellipse (in which the moon would proceed to move, if the additional force should cease) will have its line of apses regressing, while the moon is near perigee, and progressing while she is near apogee. There is, however, no incongruity between the immobility of the line of apses in the first mode of consideration, and the progress or regress in the second; because the line of apses of the instantaneous ellipse in the second case, is an imaginary line, determined by supposing the disturbing force to cease, and the moon to move undisturbed. At the apses, however, the line of apses must be the same in both methods of consideration; since, whether the disturbing force cease or not, the perpendicularity of the direction of the motion to the radius vector determines the place of an apse. Consequently, while the moon moves from one apse to the other, the motions of the line of apses in the second mode of consideration, must be such as to produce the same effect on the position of the line of apses as in the first mode of consideration; that is, they must not have altered its place; and hence the progression at one time must be exactly equal to the regression at the other time.

disturbing force directed to A in the neighbourhood of B_2 and B_4 scarcely produces any effect, as on one side of each of those points the effect is of one kind, and on the other side it is of the opposite kind, (55.)

(99.) The disturbing force directed from A, though the only one at B_1 and B_3, is not, however, the only one in the neighbourhood of B_1 and B_3. While the moon is approaching to B_1, the force perpendicular to the radius vector accelerates the moon, and therefore, by (65.), as B_1 is the place of perigee, the line of apses regresses; when the moon has passed B_1, the force retards the moon, and, therefore, by (66.), the line of apses still regresses. But when the moon is approaching B_3 the force perpendicular to the radius vector accelerates the moon, and therefore, by (65.) and (66.) as B_3 is the place of apogee, the line of apses progresses: when the moon has passed B_3 the force retards the moon and the line of apses still progresses. The question now is, whether the progression produced by the force perpendicular to the radius vector near B_3, will or will not exceed the regression produced near B_1? To answer this we must observe, that the rate of this progress or regress depends entirely upon the proportion * which the velocity produced by the disturbing force bears to the velocity of the moon; and since from B_2 to B_3 and from

* Suppose, for facility of conception, that the force perpendicular to the radius vector, acts in only one place in each quadrant between syzygies and quadratures. The portions of the orbit which are bisected by the line of syzygies will be described with greater velocity in consequence of this disturbance (abstracting all other causes) than the other portions. Now the curvature of any part of an orbit does not depend on the central force simply, or on the velocity, but on the relation between them; so that the same curve may be described either by leaving the central force unaltered and increasing the velocity in a given proportion, or by diminishing the central force in a corresponding proportion, and leaving the velocity unaltered. Consequently, in the case before us, the same curve will be described as if, without alteration of velocity, the central force were diminished, while the moon passed through the portions bisected by the line of syzygies. If now the imaginary diminution of central force were in the same proportion (that is, if the real increase of velocity were in the same proportion) at both syzygies, which here coincide with the apses, the regression of the line of apses produced at perigee, would be equal to the progression produced at apogee. But the increase of velocity produced by the force perpendicular to the radius vector near apogee, is much greater than that near perigee. First, because the force is greater, in proportion to the distance. Second, because the time of describing a given small angle is greater in proportion to the square of the distance; so that the acceleration produced while the moon passes through a given angle, is proportional to the cube of the distance. Third, because the velocity, which is increased by this acceleration, is inversely proportional to the distance; so that the ratio in which the velocity is increased is proportional to the fourth power of the distance. The effect at the greater distance, therefore, predominates over that at the smaller distance; and therefore, on the whole, the force perpendicular to the radius vector produces an effect similar to its apogeal effect; that is, it causes the line of apses to progress.

B_3 to B_4 the disturbing force is greater than that from B_4 to B_1, and from B_1 to B_2, and acts for a longer time (as by the law of equable description of areas, the moon is longer moving from B_2 to B_3 and B_4, than from B_4 to B_1 and B_2), and since the moon's velocity in passing through B_2, B_3, B_4, is less than her velocity in passing through B_4, B_1, B_2, it follows that the effect in passing through B_2, B_3, B_4, is much greater than that in passing through B_4, B_1, and B_2. Consequently, the effect of this force also is to make the line of apses progress.

(100.) On the whole, therefore, when the perigee is turned towards the sun, the line of apses progresses rapidly. And the same reasoning applies in every respect when the perigee is turned from the sun.

(101.) In the second place, suppose that the line of apses is perpendicular to the line joining the earth and sun. The disturbing force at both apses is now directed to the earth, and consequently, by (50.) and (53.), while the moon is near perigee, the disturbing force causes the line of apses to progress, and while the moon is near apogee the disturbing force causes the line of apses to regress. Here, as in the last article, the effects at perigee and at apogee would balance if the disturbing force were inversely proportional to the square of the distance from the earth. But the disturbing force is really proportional to the distance from the earth: and, therefore, as in the last article, the effect of the disturbing force while the moon is at apogee preponderates over the other; and therefore, the force directed to the centre causes the line of apses to regress.

(102.) We must also consider the force perpendicular to the radius vector. In this instance, that force retards the moon while she is approaching to each apse, and accelerates her as she recedes from it. The effect is, that when the moon is near perigee the force causes the line of apses to progress, and when near apogee it causes the line of apses to regress (65.) and (66.) The latter is found to preponderate, by the same reasoning as that in (99.) From the effect, then, of both causes the line of apses regresses rapidly in this position of the line of apses.

(103.) It is important to observe here, that the motion of the line of apses would not, as in (56.), be greater if the excentricity of the orbit were smaller. For though the motion of the line of apses is greater *in proportion to the force which* *causes it* when the excentricity is smaller; yet, in the present instance, the force which causes it is itself proportional to the excentricity (inasmuch as it is the difference of the forces at perigee and apogee, which would be equal if there were no excentricity): so that if the excentricity were made less, the force which causes the motion of the line of apses would also be made less, and the motion of the line of apses would be nearly the same as before.

(104.) It appears then, that when the line of apses passes through the sun, the disturbing force causes that line to progress; when the earth has moved round the sun, or the sun has appeared to move round the earth, so far that the line of apses is perpendicular to the line joining the sun and the earth, the line of apses regresses from the effect of the disturbing force; and at some intermediate position, it may easily be imagined that the force produces no effect on it. It becomes now a matter of great interest to inquire, whether upon the whole the progression exceeds the regression. Now the force perpendicular to the radius vector, considered in (99.), is almost exactly equal to that considered in (102.); so that the progression produced by that force when the line of apses passes through the sun, is almost exactly equal to the regression which it produces when the line of apses is perpendicular to the line joining the earth and sun; and this force may, therefore, be considered as producing no effect (except indirectly, as will be hereafter mentioned.) But the force in the direction of the radius vector, tending from the earth in (98.), is, as we have mentioned in (80.), almost exactly double of that tending to the earth in (101.), and, therefore, its effect predominates: and, therefore, on the whole, the line of apses progresses. In fact, the progress, when the line of apses passes through the sun, is about 11° in each revolution of the moon; the regress, when the line of apses is perpendicular to the line joining the earth and sun, is about 9° in each revolution of the moon.

(105.) The progression of the line of apses of the moon is considerably greater than the first consideration would lead us to think, for the following reasons.

(106.) Firstly. The earth is revolving round the sun, or the sun appears to move round the earth, in the same direction in which the moon is going. This lengthens the time for which the sun acts in any one manner upon the moon, but it lengthens it more for the time in which the moon is moving slowly, than for that in which it is moving quickly. Thus; suppose that the moon's angular motion when she is near perigee is fourteen times the sun's angular motion: and when near apogee, only ten times the sun's motion. Then she passes the sun at the former time, (as seen from the earth,) with $\frac{13}{14}$ths of her whole motion, but at the latter with only $\frac{9}{10}$ths; consequently, when near perigee, the time in which the moon passes through a given angle from the moving line of syzygies, (or the time in which the angle between the sun and moon increases by a given quantity,) is $\frac{14}{11}$ths of the time in which it would have passed through the same angle, had the sun been stationary; when near apogee, the number expressing the proportion is $\frac{10}{9}$ths. The latter number is greater than the former; and, therefore, the effect of the forces acting near apogee is increased in a greater proportion than that of the forces acting near perigee. And as the effective motion of the line of apses is produced by the excess of the apogeal effect above the perigeal effect, a very small addition to the former will bear a considerable proportion to the effective motion previously found; and thus the effective motion will be sensibly increased.

(107.) Secondly. When the line of apses is directed toward the sun, the whole effect of the force is to make it progress, that is, to move in the same direction as the sun: the sun passes through about 27° in one revolution of the moon, and, therefore, departs only 16° from the line of apses; and therefore the apse continues a long time near the sun. When at right angles to the line joining the earth and sun, the whole effect of the force is to make it regress, and therefore, moving in the direction opposite to the sun's motion, the angle between the sun and the line of apses is increased by 36° in each revolution, and the line of apses soon escapes from this position. The effect of the former force is therefore increased, while that of the latter is diminished: and the preponderance of the former is much increased. It is in increasing the rapidity of progress at one time, and the rapidity of regress at another, that the force perpendicular to the radius vector indirectly increases the effect of the former in the manner just described.

(108.) From the combined effect of these two causes the actual progression of the line of apses is nearly doubled.

(109.) The line of apses upon the whole, therefore, progresses; and (as calculation and observation agree in showing) with an angular velocity that makes it (on the average) describe 3° in each revolution of the moon, and that carries it completely round in nearly nine years. But as it sometimes progresses and sometimes regresses for several months together, its motion is extremely irregular. The general motion of the line of apses has been known from the earliest ages of astronomy.

(110.) V. For the alteration of the excentricity of the moon's orbit: first, let us consider the orbit in the position in which the line of apses passes through the sun, *fig.* 31. While the moon moves from B_1 (the perigee,) to B_3, (the apogee,) the force in the direction of the radius vector is sometimes directed to the earth, and sometimes from the earth, and therefore, by (57.) and (59.), it sometimes diminishes the excentricity and sometimes increases it. But while the moon moves from B_3 to B_1, there are exactly equal forces acting in the same manner at corresponding parts of the half-orbit, and these, by (58.), will produce effects exactly opposite. On the whole, therefore, the disturbing force in the direction of the radius vector produces no effect on the excentricity. The force perpendicular to the radius vector increases the moon's velocity when moving from B_4 to B_1, and diminishes it when moving from B_1 to B_2; in moving, therefore, from B_4 to B_1, the excentricity is increased (65.), and in moving from B_1 to B_2, it is as much diminished (66.). Similarly in moving from B_2 to B_3, the excentricity is diminished, and in moving from B_3 to B_4, it is as much increased. This force, therefore, produces no effect on the excentricity.

On the whole, therefore, while the line of apses passes through the sun, the disturbing forces produce no effect on the excentricity of the moon's orbit.

(111.) When the line of apses is perpendicular to the line joining the earth and sun, the same thing is true. Though the forces near perigee and near apogee are not now the same as in the last case, their effects on different sides of perihelion and aphelion balance each other in the same way.

(112.) But if the line of apses is inclined to the line joining the earth and sun, as in *fig.* 32., the

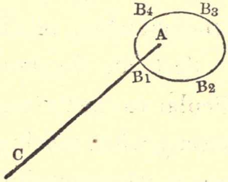


Fig. 32.

effects of the forces do not balance. While the moon is near B_4 and near B_2, the disturbing force in the radius vector is directed to the earth; at B_4 therefore, (58.), as the moon is moving towards perigee, the excentricity is increased; and at B_2, as the moon is moving from perigee, the excentricity is diminished. From the slowness of the motion at B_2, (which gives the disturbing force more time to produce its effects,) and the greatness of the force, the effect at B_2 will preponderate, and the combined effects at B_2 and B_4 will diminish the excentricity. This will appear from reasoning of the same kind as that in (98.). At B_1 and B_3, the force in the radius vector is directed from the earth: at B_1, therefore, by (59.), as the moon is moving from perigee, the excentricity is increased, and at B_3 it is diminished: but from the slowness of the motion at B_8 and the magnitude of the force, the effect at B_3 will preponderate, and the combined effects at B_1 and B_3 will diminish the excentricity. On the whole, therefore, the force in the direction of the radius vector diminishes the excentricity. The force perpendicular to the radius vector retards the moon from B_1 to B_2, but the first part of this motion may be considered near perigee, and the second near apogee, and, therefore, in the first part, it diminishes the excentricity, and in the second increases it; and the whole effect from B_1 to B_2 is very small. Similarly the whole effect from B_3 to B_4 is very small. But from B_4 to B_1, the force accelerates the moon, and therefore, by (68.), (the moon being near perigee) increases the excentricity; and from B_2 to B_3, the force also accelerates the moon, and by (68.) (the moon being near apogee) diminishes the excentricity; and the effect is much* greater

* To the reader who is acquainted with Newton's 3rd section, the following demonstration of this point will be sufficient. Four times the reciprocal of the *latus rectum* is equal to the sum of the reciprocals of the apogeal and perigeal distances. The effect of an increase of velocity at perigee in a given proportion is to alter the area described in a given time in the same proportion, and therefore, to alter the *latus rectum* in a corresponding proportion. Consequently an increase of velocity at perigee in a given proportion alters the reciprocal of the apogeal distance by a given quantity, and, therefore, alters the apogeal distance by a quantity nearly

(from the slowness of the moon and the greatness of the force) between B_2 and B_3 than between B_4 and B_1, and therefore the combined effect of the forces in these two quadrants is to diminish the excentricity.

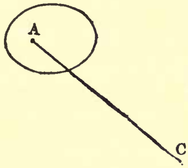
On the whole, therefore, when the line of apses is inclined to the line joining the earth and sun, in such a manner that the moon passes the line of apses before passing the line joining the earth and

proportional to the square of the apogeal distance; and, therefore, the ratio of the alteration of apogeal distance to apogeal distance (on which the alteration of excentricity depends) is nearly proportional to the apogeal distance. Similarly, if the velocity at apogee is increased in a given proportion, the ratio of the alteration of perigeal distance to perigeal distance (on which the alteration of excentricity depends) is nearly proportional to the perigeal distance. Thus if the velocity were increased in the same proportion at perigee and at apogee, the increase of excentricity at the former would be greater than the diminution at the latter, in the proportion of apogeal distance to perigeal distance. But in the case before us, the proportion of increase of velocity is much greater at apogee than at perigee. First, because the force is greater, (being in the same proportion as the distance.) Second, because the time in which the moon describes a given angle is greater, (being in the same proportion as the square of the distance,) so that the increase of velocity is in the proportion of the cube of the distance. Third, because the actual velocity is less, (being inversely as the distance,) so that the ratio of the increase to the actual velocity is proportional to the fourth power of the distance. Combining this proportion with that above, the alterations of excentricity in the case before us, produced by the forces acting at apogee and at perigee, are in the proportion of the cubes of the apogeal and perigeal distances respectively.

sun, the excentricity is diminished at every revolution of the moon.

(113.) In the same manner it will appear that

Fig. 33.



if the line of apses is so inclined that the moon passes the line of apses after passing the line joining the earth and sun, the excentricity is increased at every revolution of the moon. Here the force in the radius vector is directed to the earth, as the moon moves from perigee and from apogee: and is directed from the earth as the moon moves to perigee and to apogee; which directions are just opposite to those in the case already considered. Also the force perpendicular to the radius vector retards the moon both near perigee and near apogee; and this is opposite to the direction in the case already considered. On the whole, therefore, the excentricity is increased at every revolution of the moon.

(114.) In every one of these cases the effect is exactly the same if the sun be supposed on the side of the moon's orbit, opposite to that represented in the figure.

(115.) Now the earth moves round the sun, and the sun therefore appears to move round the earth in the order successively represented by the *figs*. 31, 32, and 33. Hence then; when the sun is in the line of the moon's apses, the excentricity does not alter (110.); after this it diminishes till the sun is seen at right angles to the line of apses (112.); then it does not alter (111.): and after this it increases till the sun reaches the line of apses on the other side. Consequently, the excentricity is greatest when the line of apses passes through the sun; and is least when the line of apses is perpendicular to the line joining the earth and sun.

The amount of this alteration in the excentricity of the moon's orbit is more than $\frac{1}{5}$th of the mean value of the excentricity; the excentricity being sometimes increased by this part, and sometimes as much diminished; so that the greatest and least excentricities are nearly in the proportion of 6 : 4 or 3 : 2.

(116.) The principal inequalities in the moon's motion may therefore be stated thus:

1st. The *elliptic inequality*, or *equation of the centre* (31.), which would exist if it were not disturbed.

2nd. The *annual equation* (90.), depending on the position of the earth in the earth's orbit.

3rd. The *variation* (93.), and *parallactic inequality* (94.), depending on the position of the moon with respect to the sun.

4th. The general *progression of the moon's perigee* (104.)

5th. The *irregularity in the motion of the perigee,* depending on the position of the perigee with respect to the sun (109.)

6th. The *alternate increase and diminution of the excentricity,* depending on the position of the perigee with respect to the sun (115.)

These inequalities were first explained (some imperfectly) by Newton, about A. D. 1680.

(117.) The effects of the two last are combined into one called the *evection.* This is by far the largest of the inequalities affecting the moon's place: the moon's longitude is sometimes increased 1° 15′ and sometimes diminished as much by this inequality. It was discovered by Ptolemy, from observation, about A. D. 140.

(118.) It will easily be imagined that we have here taken only the principal inequalities. There are many others, arising chiefly from small errors in the suppositions that we have made. Some of these, it may easily be seen, will arise from variations of force which we have already explained. Thus the difference of disturbing forces at conjunction and at opposition, whose principal effect was discussed in (94.), will also produce a sensible inequality in the rate of progression of the line of apses, and in the dimensions of the moon's orbit. The alteration of disturbing force depending on the excentricity of the earth's orbit will cause an alteration in the magnitude of the *variation* and the *evection*. The alteration of that part mentioned in (94.) produces a sensible effect depending on the angle made by the moon's radius vector with the earth's line of apses. All these, however, are very small: yet not so small but that, for astronomical purposes, it is necessary to take account of thirty or forty.

(119.) There is, however, one inequality of great historical interest, affecting the moon's motion, of which we may be able to give the reader a general idea. We have stated in (89.) that the effect of the disturbing force is, upon the whole, to diminish the moon's gravity to the earth: and in (90.) we have mentioned that this effect is greater when the earth is near perihelion, than when the earth is near aphelion. It is found, upon accurate investigation, that half the sum of the effects at perihelion and at aphelion is greater than the effect at mean distance, by a small quantity depending on the excentricity of the earth's orbit: and, consequently, the greater the excentricity (the mean distance being unaltered) the greater is the effect of the sun's disturbing force. Now, in the lapse of ages, the earth's mean distance is not sensibly altered by the disturbances which the planets produce in its motion; but the excentricity of the earth's orbit is sensibly diminished, and has been diminishing for thousands of years. Consequently the effect of the sun in disturbing the moon has been gradually diminishing, and the gravity to the earth has therefore, on the whole, been gradually increasing. The size of the moon's orbit has therefore, gradually, (but insensibly,) diminished (47.): but the moon's place in its orbit has sensibly altered (49.), and the moon's angular motion has appeared to be perpetually quickened. This phænomenon was known to astronomers by the name of the *acceleration of the moon's mean motion,* before it was theoretically explained in 1787, by Laplace: on taking it into account, the oldest and the newest observations are equally well represented by theory. The rate of progress of the moon's line of apses has, from the same cause, been somewhat diminished.

SECTION VI.—*Theory of Jupiter's Satellites.*

(120.) JUPITER has four satellites revolving round him in the same manner in which the moon revolves round the earth; and it might seem, therefore, that the theory of the irregularities in the motion of these satellites is similar to the theory of the irregularities in the moon's motion. But the fact is, that they are entirely different. The fourth satellite (or that revolving in the largest orbit) has a small irregularity analogous to the moon's variation, a small one similar to the evection, and one similar to the annual equation: but the last of these amounts only to about two minutes, and the other two are very much less. The corresponding inequalities in the motion of the other satellites are still smaller. But these satellites disturb each other's motions, to an amount and in a manner of which there is no other example in the solar system; and (as we shall afterwards mention) their motions are affected in a most remarkable degree by the shape of Jupiter.

(121.) The theory, however, of these satellites is much simplified by the following circumstances:— First, that the disturbances produced by the sun may, except for the most accurate computations, be wholly neglected. Secondly, that the orbits of the two inner satellites have no excentricity independent of perturbation. Thirdly, that a very remarkable relation exists (and, as we shall show, necessarily exists) between the motions of the three first satellites.

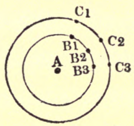
Before proceeding with the theory of the first three satellites, we shall consider a general proposition which applies to each of them.

(122.) Suppose that two small satellites revolve round the same planet; and that the periodic time of the second is a very little greater than double the periodic time of the first; what is the form of the orbit in which each can revolve, describing a curve of the same form at every revolution?

(123.) The orbits will be sensibly elliptical, as the perturbation produced by a small satellite in one revolution will not sensibly alter the form of the orbit. The same form being supposed to be described each time, the major axis and the excentricity are supposed invariable, and the position of the line of apses only is assumed to be variable. The question then becomes, What is the excentricity of each orbit, and what the variation of the position of the line of apses, in order that a curve of the same kind may be described at every revolution?

(124.) In *fig.* 34. let B_3, B_1, B_2, represent the orbit of the first, and C_3, C_1, C_2, the orbit of the second. Suppose that when B was at B_1, C

Fig. 34.



was at C_1, so that A, B_1, C_1, were in the same straight line, or that B and C were in conjunction at these points. If the periodic time of C were exactly double of the periodic time of B, B would have made exactly two revolutions, while C made exactly one; and, therefore, B and C would again be in conjunction at B_1 and C_1. But as the periodic time of C is a little longer than double that of B, or the angular motion of C rather slower than is supposed, B will have come up to it (in respect of longitude as seen from A) at some line B_2 C_2, which it reaches *before* reaching the former line of conjunction $B_1 C_1$. And it is plain that there has been no other conjunction since that with which we started, as the successive conjunctions can take place only when one satellite has gained a whole revolution on the other. The first conjunction then being in the line $A B_1 C_1$, the next will be in the line $A B_2 C_2$, the next in a line $A B_3 C_3$, still farther from the first, &c.; so that the line of conjunction will regress slowly; and the more nearly the periodic time of one satellite is double that of the other, the more slowly will the line of conjunction regress.

(125.) As the principal part of the perturbation is produced when the satellites are near conjunction, (in consequence of the smallness of their distance at that time,) it is sufficiently clear that the position of the line of apses, as influenced by the perturbation, must depend on the position of the line of conjunction; and, therefore, that the motion of the line of apses must be the same as the motion of the line of conjunction. Our question now becomes this: What must be the excentricities of the orbits, and what the positions of the perijoves, in order that the motions of the lines of apses, produced by the perturbation, may be the same as the motion of the line of conjunction?

(126.) If the line of apses of the first satellite does not coincide with the line of conjunction, the first satellite at the time of conjunction will either be moving from perijove towards apojove, or from apojove towards perijove. If the former, the disturbing force, which is directed from the central body, will, by (59.), cause the excentricity to increase; if the latter, it will cause it to decrease. As we have started with the supposition, that the excentricity is to be supposed invariable, neither of these consequences can be allowed, and, therefore, the line of apses must coincide with the line of conjunction.

(127.) If the apojove of the first satellite were in the direction of the points of conjunction, the disturbing force in the direction of the radius vector, being directed from the central body, would, by (54.), cause the line of apses to progress. Also the force perpendicular to the radius vector, before the first satellite has reached conjunction, (and when the second satellite, which moves more slowly, is nearer to the point of conjunction than the first,) tends to accelerate the first satellite; and that which acts after the satellites have passed conjunction, tends to retard the first satellite; and both these, by (65.) and (66.), cause the line of apses to progress. But we have assumed, that the line of apses shall move in the same direction as the line of conjunction, that is, shall regress; therefore, the apojove of the first satellite cannot be in the direction of the points of conjunction.

(128.) But if we suppose the perijove of the first satellite to be in the direction of the points of conjunction, every thing becomes consistent. The disturbing force, in the direction of the radius vector, from the central body, will, by (51.), cause the line of apses to regress. The force perpendicular to the radius vector, which accelerates the first satellite before it has reached conjunction, that is, before it has reached the perijove, and retards it after that time, will also, by (65.) and (66.), cause the line of apses to regress. Also, as in (56.), this regression will be greater as the excentricity of the orbit is less, because the disturbing force, which acts here, does not depend on the excentricity. By proper choice, therefore, of a value of the excentricity, we can make an orbit, whose line of apses will always regress exactly as fast as the line of conjunction, and will, therefore, always coincide with it; whose excentricity, in consequence, will never alter, by (59.) and

(68.); and whose general shape, therefore, will be the same at every successive revolution.

(129.) We shall mention hereafter, that the form of Jupiter is such as would cause the perijove of the first satellite, if it were not disturbed by the second satellite, to progress with a velocity not depending upon the excentricity of the orbit. The only alteration which this makes in our conclusions is, that the excentricity of the orbit must be so chosen, that the perturbation of which we have spoken will cause a regression equal to the *sum* of the progression which Jupiter's shape would occasion, and the regression of the line of conjunction. As this is greater than the regression of the line of conjunction alone, the excentricity of the orbit must be less. So that the only effect of Jupiter's shape is to diminish, in some degree, the excentricity of the orbit.

(130.) Now let us inquire what must be the form and position of the orbit of the second satellite. As before, the principal part of the perturbation is near conjunction. At and near the conjunction, the disturbing force, in the direction of the radius vector, is directed to the central body. Before conjunction, when the first satellite is less advanced than the second, the disturbing force, perpendicular to the radius vector, retards the second, by (86.). For, the periodic time of the second being nearly double that of the first, the mean distances from the planet will be nearly in the proportion of 7 to 11, (as the proportion of the cube of 7 to the cube of 11 is nearly the same as the proportion of the square of 1 to the square of 2, see (34.),) and, therefore, near conjunction, the distance of the first from the second is less than the distance of the first from the central body. After conjunction, the disturbing force accelerates the second body. Now, without going through several cases as before, which the reader will find no trouble in doing for himself, we shall remark, at once, that if the apojove of the second satellite is in the direction of the points of conjunction, both the disturbing force, directed to the central body at apojove, and that perpendicular to the radius vector, retarding it before it reaches apojove, and accelerating it afterwards, by (53.), (65.), and (66.), will cause the line of apses to regress; and that, by proper choice of excentricity, the regression of the line of apses may be made exactly equal to the regression of the line of conjunction.

(131.) Our conclusion, therefore, is: If two satellites revolve round a primary, and if the periodic time of one is very little greater than double the periodic time of the other, and if we assume that the orbits described have always the same form; (that is, if they have no excentricity independent of perturbation;) then the orbits will not sensibly differ from ellipses, the lines of apses of both orbits must always coincide with the line of conjunctions, and the perijove of the first orbit, and the apojove of the second, must always be turned towards the points of conjunction. It appears also, that these conditions are sufficient, inasmuch as the rate of regress of the lines of apses will (with proper values for the excentricities) be the same as the rate of regress of the line of conjunctions, and the excentricities then will not change. The excentricities of the orbits will be greater as the regress of the line of conjunctions is slower, or as the proportion of the periodic times approaches more exactly to the proportion of 1 : 2.

(132.) In the same manner it would be found, that if the periodic time of one satellite were very little less than double that of the other, the lines of apses (in order that similar orbits may be traced out at each revolution) must always coincide with the line of conjunction, and the apojove of the first satellite and the perijove of the second must always be turned towards the points of conjunction ; and the excentricities of the orbits must be greater, as the proportion of the periodic times approaches more exactly to the proportion of 1 : 2.

(133.) The same thing exactly would hold, if the periodic times were very nearly in the ratio of 2:3, or of 3 : 4, &c., but these suppositions do not apply to Jupiter's satellites.

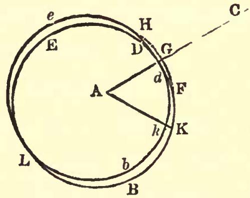
(134.) Having thus found the distortion produced by the disturbing force in orbits which have no excentricity, independent of perturbation, it will easily be imagined, that the same kind of distortion will be produced if the orbits have an original excentricity. If we make, in an elliptic orbit, the same kind of alteration which must be made in a circular orbit, in order to form the figure found above, we shall have nearly the orbit that will be described from the combined effects of perturbation and of excentricity independent of perturbation *.

* The truth of this proposition may be shown more fully in the following manner:—Let A, *fig.* 35, be the place of the primary, A C the line of conjunctions of the first and second satellite, B D E the elliptic orbit, in which the first satellite would move if undisturbed, D its perijove. Suppose (to simplify the figure) that the attraction of the second satellite acts only for a limited space;

We shall now proceed with the application of these conclusions to Jupiter's first three satellites.

for instance, while the first satellite passes from F to H. Then the result of the investigations from (122.) to (131.) is, that the

Fig. 35.

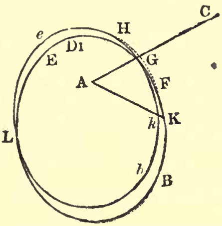


first satellite will be drawn outwards from the orbit in which it would have moved, so as to describe a curve F G H; and when the disturbing force ceases at H, it will proceed to describe an ellipse, H *e b d*, similar to B D E, but with this difference, that the perijove is at *d* instead of D. The conclusion, however, now that it has been securely obtained from the reasoning above, may be stated as the result of the following reasoning:—In consequence of the disturbing force, which has drawn the first satellite outwards, without, upon the whole, altering its velocity, (accelerating it before conjunction, and retarding it afterwards,) the satellite has moved in a curve, F G H, external to the ellipse F D, in which it would have moved; and after the disturbing force has ceased at H, the satellite (which is moving in a path inclined externally from the old orbit) continues to recede from the old orbit till the diminution of velocity (26.) allows its path to be so much curved, that at *e* it begins to approach, and at L the new orbit intersects the old one; and after this, the path is inclined internally from the old orbit, till the increase of velocity (25.) makes its path so little curved that it approaches the old orbit again, and again crosses it between *d* and D. In like manner, if, as in *fig.* 36., the orbit B F E have an excentricity independent of perturbation, (the perijove being at any point D′,) nevertheless, we may state that, in consequence of the disturbing force, the

(135.) The periodic time of Jupiter's first satellite is, 1 day, 18 hours, 27 minutes, and 34

satellite will move in a curve F G H external to F E; but when the disturbing force ceases at H, the satellite (which is moving in

Fig. 36.



a path inclined externally from the old orbit) continues to recede from the old orbit till the diminution of velocity (26.) allows its path to be so much curved, that it begins to approach at some point *e;* that at some point L, nearly opposite to C, the new orbit intersects the old one; and that, after this, the path is inclined internally from the old orbit, till the increase of velocity (25.) makes its path so little curved that it approaches the old orbit again, and again crosses it between F and H. Thus, the alteration of the radius vector, drawn in any given direction, as A K (which in the new orbit is altered to A *k*) is nearly the same in the second case as in the first. This, however, is the alteration produced in a single revolution of the satellite; but as the same applies to every successive revolution, it follows that the inequality or variation of the radius vector in the second case is nearly the same as in the first case; and thus the proposition of the text is proved.

The inequality of the radius vector would be somewhat different if the excentricity of the orbit in the second case were considerable, partly because the places of conjunction would not be at equal angular distances, partly because the disturbing forces would be different, (as the distance between the satellites in conjunction would not always be the same,) and partly because the effect of a given force is really different, according to the part of the orbit at seconds; that of the second satellite is 3 days, 13 hours, 13 minutes, and 42 seconds; that of the third satellite is, 7 days, 3 hours, 42 minutes, and 32 seconds. The periodic time of the second satellite exceeds, by a small quantity, double that of the first, so that the preceding investigations apply

which it acts. But where the excentricity is so small, as in the orbit of Jupiter's third satellite, or in those of the old planets, the alteration of the inequality of the radius vector produced by these differences is hardly sensible.

The reasoning of this note may be applied, with the proper alterations, to every case of perturbation, produced by a disturbing force which is nearly independent of the form of the orbit; and as this will apply successively to each of the causes producing disturbance, we shall at last arrive at the following general proposition:—If several disturbing forces act on a planet or satellite, and if we estimate the inequality in the radius vector, which each of these would produce, supposing the orbit to have no excentricity independent of perturbation; then the inequality really produced, supposing the orbit to have an independent excentricity, will be nearly the same as the sum of all the inequalities so estimated.

It is to be remarked, that if an orbit have an independent excentricity, and if the orbit receive an alteration similar to an elliptic inequality, (that is, if it be elongated on one side and flattened on the other,) the orbit is still sensibly an ellipse, of which the original focus is still the focus. Thus, in the instance occupying the first part of this note, as the inequality impressed on the elliptic orbit in the second case is the same as the inequality in the first case, that is, is similar to an elliptic inequality, the orbit so altered will still be an ellipse, whose excentricity and line of apses are altered. We might, therefore, have obtained our results by at once investigating the alterations of the excentricity and line of apses produced by the disturbing forces; but the method adopted in the text is simpler.

to the motion of these two satellites. In fact, 275 revolutions of the first satellite are finished in almost exactly the same time as 137 revolutions of the second. If then, at a certain time, these two satellites start from conjunction, they will be in conjunction near the same place at every revolution of the second satellite, or at every second revolution of the first satellite: but the line of conjunction will regress slowly; and when the first satellite has finished 275 revolutions, or one revolution more than double the number made by the second satellite, they will again be in conjunction in the same place as before, the line of conjunction having regressed till it has again reached the same position: this takes place in 486½ days.

(136.) From the preceding investigation then it appears that, as these orbits have no excentricity independent of perturbation, they will be elliptic, and the line of apses of each orbit will regress so as to turn completely round in 486½ days; and that when in conjunction, the first satellite will always be in perijove, and the second satellite will always be in apojove.

(137.) But the periodic time of the third satellite is almost exactly double that of the second satellite, exceeding the double by a small quantity;

and on this account the orbit of the second satellite will be distorted from the form which otherwise it would have had, by an inequality similar to that just investigated. In a word, the line of conjunction of the second and third satellites will slowly regress, and the orbit of the second satellite will always be compressed on the side next the points of conjunction, and elongated on the opposite side; and the orbit of the third satellite will always be elongated on the side next the points of conjunction, and compressed on the opposite side.

(138.) Now we come to the most extraordinary part of this theory. We have remarked that 275 revolutions of the first satellite are finished in almost exactly the same time as 137 revolutions of the second; but it will also be found that 137 revolutions of the second are finished in almost exactly the same time as 68 revolutions of the third: all these revolutions occupying $486\frac{1}{2}$ days. Because 275 exceeds the double of 137 by 1, we have inferred that the line of conjunctions of the first and second satellites regresses completely round in 275 revolutions of the first satellite, or in $486\frac{1}{2}$ days. In like manner, because 137 exceeds the double of 68 by 1, we infer that the line of conjunctions of the second and third satellites regresses completely round in 137 revolutions of the second satellite, or in $486\frac{1}{2}$ days. Hence we have this remarkable fact: *the regression of the line of conjunction of the second and third satellites is exactly as rapid as the regression of the line of conjunction of the first and second satellites.* So accurate is this law, that in the thousands of revolutions of the satellites, which have taken place since they were discovered, not the smallest deviation from it (except what depends upon the elliptic form of the orbit of the third satellite) has ever been discovered.

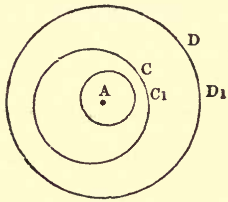
(139.) Singular as this may appear, the following law is not less so. *The line of conjunction of the second and third satellites always coincides with the line of conjunction of the first and second satellites produced backwards, the conjunctions of the second and third satellites always taking place on the side opposite to that on which the conjunctions of the first and second take place.* This defines the relative position of the lines of conjunction, which (by the law of last article) is invariable. Like that law it has been found, as far as observation goes, to be accurately true in every revolution since the satellites were discovered.

(140.) The most striking effect of these laws in the perturbations of the satellites is found in the motions of the second satellite. In consequence of the disturbing force of the first satellite, the orbit of the second satellite will be elongated towards the points of conjunction of the first and second (130.), and consequently compressed on the opposite side. In consequence of the disturbing force of the third satellite, the orbit of the second satellite will be compressed on the side next the points of conjunction of the second and third (128.) And because the points of conjunction of the second and third are always opposite to the points of conjunction of the first and second, the place of compression from one cause will always coincide with the place of compression from the other cause; and therefore, the orbit of the second satellite will be very much compressed on that side, and consequently very much elongated on the other side. The excentricity of the orbit, depending thus entirely on perturbation, exceeds considerably the excentricity of the orbit of Venus. The inequalities in the motions of the satellites, produced by these excentricities, were first discovered (from observation) by Bradley about A. D. 1740, and first explained from theory by Lagrange, in 1766.

(141.) The singularity of these laws, and the accuracy with which they are followed, lead us to suppose that they do not depend entirely on chance. It seems natural to inquire whether some reason may not be found in the mutual disturbances of the satellites, for the preservation of such simple relations. Now we are able to show that, supposing the satellites put in motion at any one time, nearly in conformity with these laws, their mutual attraction would always tend to make their motions follow these laws exactly. We shall show this by supposing a small departure from the law, and investigating the nature of the forces which will follow as a consequence of that departure.

(142.) Suppose, for instance, that the third satellite lags behind the place defined by this law; that is, suppose that, when the second satellite is at the most compressed part of its ellipse, (as produced by the action of the first satellite,) the third satellite is behind that place. The conjunction then of the second and third satellites will happen before reaching the line of apses of the orbit of the second, as produced by the action of the first. Now in the following estimation of the forces which act on the third satellite, and of their variation depending on the variation of the positions of the lines of conjunction, there is no need to consider the influence which the ellipticity of the orbit of the second as produced by the third, or that of the third as produced by the second, exerts upon the third satellite; because the flattening arising from the action of the third, and the elongation arising from the action of the second, will always be turned towards the place of conjunction of the second and third, and the modification of the action produced by this flattening and elongation will always be the same, whether the lines of conjunction coincide or not. In *fig.* 37., let C be the perijove of the orbit

Fig. 37.



of the second satellite, (as produced by the action of the 1st satellite alone,) D the point of the orbit of the third which is in the line A C produced. If the third satellite is at D when the second is at C, the force produced by the second perpendicular to the radius vector, retards the third before it reaches D, and accelerates it after it has passed D, by equal quantities. But if, as in the supposition which we have made, the conjunction takes place in the line A C_1 D_1, the retardation of the third satellite before conjunction is produced by the attraction of the second satellite before it arrives at perijove, when it is near to the orbit of the third satellite, (and therefore acts powerfully,) and moves slowly, (and therefore acts for a long time;) while the acceleration after conjunction is produced by the second satellite near its perijove, when it is far from the orbit of the third satellite, (and therefore acts weakly,) and moves rapidly (and therefore acts for a short time). The retardation therefore exceeds the acceleration; and the consequence is, by (48.), that the periodic time of the third satellite is shortened, and therefore its angular motion is quickened; and therefore, at the next conjunction, it will have gone further forward before the second satellite can come up with it, or the line of conjunction will be nearer to the place of perijove of the second satellite, depending on the action of the first. In the same manner, if we supposed the third satellite moving rather quicker than it ought in conformity with the law, the tendency of the forces would be to accelerate it, to make its periodic time longer, and thus to make its angular motion slower. By the same kind of reasoning it will be seen that there are forces acting on the first satellite, produced by the elliptic inequality which the third impresses on the orbit of the second, tending to accelerate the angular motion of the first satellite in the first case, and to retard it in the second. The same reasoning will also show that both the first and third satellite exert forces on the second, tending to retard its angular motion in the first case, and to accelerate it in the second. All these actions tend to preserve the law: in the first case by making the line of conjunctions of the first and second satellite regress, and that of the second and third progress, till they coincide; and in the second case, by altering them in the opposite way, till they coincide.

(143,) Perhaps there is no theoretical permanence of elements on which we can depend with so great certainty, as on the continuance of this law. The greatest and most irregular perturbations of Jupiter or of his satellites, provided they come on gradually, will not alter the relation between their motions; the effect of a resisting medium will not alter it; though each of these causes would alter the motions of all the satellites; and though similar causes would wholly destroy the conclusions which mathematicians have drawn as to the stability of the solar system, with regard to the elements of the planetary orbits. The physical explanation of this law was first given by Laplace, in A. D. 1784.

(144.) We have terminated now the most remarkable part of the theory of these satellites. There are, however, some other points which are worth attending to, partly for their own sake, and partly as an introduction to the theory of the planets.

(145.) The orbit of the third satellite, as we have mentioned, has a small excentricity independent of perturbation. Consequently, when the conjunction with the second takes place near the independent perijove of the third, the effect of the disturbance on the second is rather greater than at any other time; and this produces an irregularity in the excentricity of the second, and in the motion of its apses, depending on the distance of the line of conjunction from the independent perijove of the third. The departure from uniformity in the angular motion of the third, also produces a departure from uniformity in the regression of the line of conjunction, and this contributes to the same irregularity.

(146.) The disturbing force in the direction of the radius vector, produced by an inner satellite, is sometimes directed to the central body and sometimes from it, but, on the whole, the former exceeds the latter. (86.) Now the principal part of the effect really takes place when the satellites are near conjunction; consequently, when the line of conjunction passes near the independent perijove of the third satellite, the force by which the third satellite is urged to the planet is greater than at any other time; and as the line of conjunction revolves, the force alternately increases and diminishes. This produces an irregularity in the major axis, and consequently in the motion of the third satellite (47.), depending on the distance of the line of conjunction from the perijove of the third.

(147.) The disturbing force in the direction of the radius vector produced by an outer satellite is sometimes directed to the central body, and sometimes from it, but on the whole, the latter exceeds the former. (80.) For the reasons, therefore, in the last article, there is in the motion of the second satellite an irregularity depending on the distance of the line of conjunction from the independent perijove of the third, but opposite in its nature to that of the third satellite.

(148.) Each of these irregularities in the motion of one of these satellites produces an irregularity in the motion of the others; and thus the whole theory becomes very complicated when we attempt to take the minute irregularities into account.

(149.) The motion of the fourth satellite is not related to the others in the same way in which they are related among themselves. Its periodic time is to the periodic time of the third nearly in the proportion of 7 : 3. Some of the irregularities then which it experiences and which it occasions are nearly similar to those in the motions of the planets. These, however, are small; the most important are those depending on the changes in the elements which require many revolutions of the satellites to go through all their various states, but which, nevertheless, have been observed since the satellites were discovered. We shall proceed with these.

(150.) First, let us suppose that the third satellite has no excentricity independent of perturbation, and that the fourth satellite has a sensible excentricity, its line of apses progressing very slowly, in consequence principally of the shape of Jupiter, (so slowly as not to have gone completely round in eleven thousand revolutions of the satellite.) When each of the satellites has revolved a few hundred times round Jupiter, their conjunctions will have taken place almost indifferently in every part of their orbits. If the orbit of the fourth as well as that of the third had no independent ellipticity, there would be no remarkable change of shape produced by perturbation, as the action of one satellite upon the other would be the same when in conjunction in all the different parts of the orbit. But the orbit of the fourth being excentric, the action of each satellite on the other is greatest when the conjunction happens near the perijove of the fourth satellite. We may consider then that the preponderating force takes place at this part of the orbits; and we have to inquire what form the orbit of the third satellite must have, to preserve the same excentricity at every revolution. It must be remembered here that the effect of Jupiter's shape is to cause a more rapid progress of the line of apses of the third satellite, if its orbit be excentric, than of the line of apses of the fourth.

(151.) Considering then that the preponderating force on the third satellite in the direction of the radius vector is directed from the central body towards the perijove of the fourth, and that the preponderating force perpendicular to the radius vector accelerates it as it approaches that part, and retards it afterwards, it is plain from (51.) (65.) and

(66.) that, if the perijove of the third satellite were in that position, the forces would cause the line of apses to regress; and this regression, if the excentricity of the third be small, may be considerable, (though the preponderance of force which causes it is extremely small,) and may overcome so much of the progression caused by Jupiter's shape, as to make the real motion of the line of apses as nearly equal as we please to the motion of the line of apses of the fourth. But the motion of the line of apses of the fourth will itself be affected (though very little) by the greater action of the third satellite on it at the same place; and the part in the radius vector being directed at its perijove to the central body, and the part perpendicular to the radius vector retarding it before it reaches the perijove, and accelerating it afterwards, will cause a small increase of progression of its apse. The state of things will be permanent, so far as depends on these forces, when the increased progression of the apse of the fourth satellite is equal to the diminished progression of the apse of the third; and thus the progression of the apse of the fourth will be somewhat increased, and the third satellite's orbit will have a compression corresponding in direction to the perijove of the fourth, and an elongation in the same direction as the apojove of the fourth. This would be the case if the third satellite had no excentricity independent of perturbation; but we may, as in other cases, consider that the same kind of distortion will be produced in the orbit if it has an independent excentricity.

(152.) Now let us suppose the fourth satellite to have no excentricity independent of perturbation, and the third satellite to have an independent excentricity. The greatest action will now be at the apojove of the third satellite, and this will (though in a small degree) cause the line of apses of the third satellite to progress; that is, it will increase the rapidity of progression which Jupiter's shape gives it. If, now, we wish to discover the form of orbit of the fourth satellite which will at every revolution preserve the same excentricity, and have its line of apses always corresponding with that of the third satellite, and therefore progressing more rapidly than the shape of Jupiter alone would make it progress, we must evidently suppose the perijove of the fourth satellite turned towards the apojove of the third, and, by supposing the excentricity small enough, the progression may be made as rapid as we please. Thus the effect of excentricity in the orbit of the third satellite is, that its line of apses is made to progress rather more rapidly, and that the orbit of the fourth satellite is compressed on the side next the apojove of the third satellite, and elongated on the opposite side. We have supposed for this investigation that the fourth satellite had no excentricity independent of perturbation, but the conclusion as to the distortion of the orbit may be applied if we suppose it to have independent excentricity.

(153.) In fact, the orbits of both the third and fourth satellites have independent excentricities, and both our conclusions apply to them. The fourth satellite, besides its independent excentricity, has an excentricity impressed upon it, opposite in kind to that of the third; and the third satellite, besides its independent excentricity, has an excentricity impressed upon it of the same kind as that of the fourth. In the same manner, the orbits of the first and second satellites have small excentricities impressed on them, similar in their kind to those of the third and fourth.

(154.) It will readily be conceived that the excentricities of the orbit of the third satellite will affect the great inequality (137.) which it produces in the motion of the second; and on the contrary, that the inequality in the motion of the third produced by the attraction of the second, will influence the effect of the third on the fourth. We shall not, however, notice these further than to state that their effects are small.

(155.) We have now gone over the principal inequalities of the motions of Jupiter's satellites. They are so much connected, and (as we may say) so completely entangled, that though they may be explained in the way in which we have considered them, it would hardly be possible to calculate them in that way. A mathematical process of the most abstruse kind, which will at the same time embrace the motions of all, is alone competent to this object. We shall, however, have attained our end if we have given the reader a general idea of the explanation of disturbances in the most curious and complicated system that has ever been reduced to calculation.

Section 7.—*Theory of Planets.*

(156.) The theory of the planets may be considered as holding a middle place between that of our moon and that of Jupiter's satellites. In our moon, the principal inequalities are those that exhibit themselves in nearly the same order at every revolution, or, at longest, in the earth's revolution round the sun, depending entirely upon the relative position of the moon, the sun, and the lines of apses. In Jupiter's satellites, some of the principal inequalities (as those of the third and fourth satellites) do not depend at all upon the relative position of the bodies, but depend on the position of the lines of apses, whose revolutions, though slow, may yet be completely observed. But in the planets, the terms analogous to those which we have mentioned in the moon's motions are small: the changes of elements are so slow, that, though they may be in some degree observed, many thousands of years would be necessary to observe them completely. The most remarkable irregularities are those produced by changes in the elements occupying several revolutions of the planets, and more nearly analogous to the mutual perturbations of the three first satellites of Jupiter than to any other that we have seen; differing from them, however, in this respect, that for most of them independent excentricities are quite essential.

(157.) There are, however, some terms very nearly similar to those mentioned in the theory of the moon. Suppose, for instance, we consider the perturbations of Mercury by Jupiter (whose distance from the sun is more than thirteen times as great.) This case is almost exactly analogous to the case of the moon disturbed by the sun. And in consequence, Mercury's orbit is flattened a little on the sides nearest to and farthest from Jupiter; but this effect is much disguised by the effect of forces analogous to those mentioned in (94.), which here preponderates greatly: his line of apses progresses a little at every revolution, when Jupiter is nearly in that line, and regresses a little when Jupiter is in the line perpendicular to it: his orbit is a little more excentric in the former case, and a little less so in the latter; and his orbit is a little larger when Jupiter is at perihelion than when at aphelion. The same thing applies very nearly to the disturbances of Venus, the Earth, and Mars, produced by Jupiter.

(158.) The instance taken above is almost an extreme one. When we consider the perturbations of two planets which are nearer to each other, we are obliged to alter our conclusions considerably. The disturbing force becomes so much greater where the planets are near conjunction than at any other part, that the orbit is much more changed there than at any other part. However, the reasoning upon which, in (91.), we determined the form of the moon's orbit, laying aside the consideration of independent excentricity, will, to a certain extent, apply here. The orbit in several cases will be flattened on the side where conjunction takes place, and on the opposite side, but generally most so on the latter; and will be made protuberant at the parts where the disturbing force tends wholly to increase the gravitation towards the sun. The same general reasoning will, in many cases, help us to find the form of the orbit which is influenced by the attraction of an interior planet.

(159.) A consideration, however, of particular cases will show how cautious we must be in applying this conclusion. Suppose, for instance, we consider the reciprocal perturbations of the Earth and Mars. The periodic time of Mars is nearly double that of the Earth. Here, then, we fall upon an inequality of such a kind as that discussed in (122.), &c., for the satellites of Jupiter. And though the periodic time of Mars is not *very* nearly double that of the Earth, so that the distortions produced in the orbits of the Earth and Mars are not very striking; still they are the greatest (of those depending only on the position of the planets) which these two bodies produce in each other's motions. Here, then, the disturbance, which on a hasty view we might suppose analogous to the *variation* of the Moon, becomes, from the small disproportion of distances, and the near commensurability of the periodic times, much more nearly similar to the slow variation of the elements of orbits.

(160.) It seems quite hopeless to attempt to give a notion of the calculations by which, in all the different cases, the disturbances independent of the excentricities can be computed. It is sufficient to state, that the same methods apply to all, and that they are much more simple than those relating to other points, of which an idea may be given by general explanation.

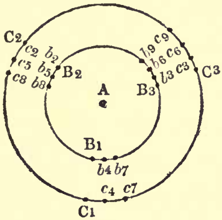
(161.) Let us now consider the inequalities of motion which depend on the excentricities and inclinations of the planets' orbits. The idea that will probably first occur to the reader is this. "If the disturbances of the planets, supposing their orbits to have no independent excentricities, amount only to a few seconds, how is it likely that the small alterations of place, which are produced by the trifling excentricities and inclinations of their orbits, will so far alter their forces upon each other as to produce any sensible difference in the magnitude of the irregularities?" In answer to this we must say, " It is true that these forces, or alterations of forces, are exceedingly small, and those parts of them which act in the same direction for a short time only (as for a fraction of the periodic time of a planet) do not produce any sensible effect. But we can find some parts of them which act in the same manner during many revolutions: the effects of these may grow up in time to be sensible; and those in particular which alter the mean distance and the periodic time may produce in time an effect on the longitude of the planet (49.), very much more conspicuous than that in the alteration of the orbit's dimensions."

(162.) In this consideration is contained the whole general theory of those inequalities known by the name of *inequalities of long period.* They are the only ones depending on the excentricities (besides those similar to the moon's evection) which ever become important.

(163.) To enter more minutely into the explanation, let us take the instance of the long inequality of Jupiter and Saturn: the most remarkable for its magnitude, and for the length of time in which the forces act in the same manner, as well as for the difficulty which it had given to astronomers before it was explained by theory, that has been noticed since the first explanation of the Moon's irregularities.

(164.) The periodic times of Jupiter and Saturn are very nearly in the proportion of 2 to 5, (the periodic times being 4332 days, 17 hours, and 10,759 days, 5 hours,) or the number of degrees of longitude that they will describe in the same time, omitting all notice of their excentricities, will be in the proportion of 5 to 2 nearly. Suppose, now, that they were exactly in the proportion of 2 to 5; and suppose that Jupiter and Saturn started from conjunction; when Saturn has described 240 degrees, Jupiter will have described 600 degrees (as these numbers are in the proportion of 2 to 5): but as 360 degrees are the circumference, Jupiter will have gone once round, and will besides have described 240 degrees. It will, therefore, again be in conjunction with Saturn. When Saturn has again described 240 degrees, that is, when Saturn has described in all 480 degrees, or has gone once round and has described 120 degrees more, Jupiter will have described 1200 degrees, or will have gone three times round and described 120 degrees more, and, therefore, will again be in conjunction with Saturn. When Saturn has again described 240 degrees, that is, when it has gone exactly twice round, Jupiter will have gone exactly five times round, and they will again be in conjunction. So that, if the periodic times were exactly in the proportion of 2 to 5, there would be a continual succession of conjunctions at the points whose longitudes exceeded the longitude of the first place of conjunction by 240°, 120°, 0, 240°, 120°, 0°, &c. Thus, in *fig.* 38., if B_1 is the place

Fig. 38.



of Jupiter at first, and C_1 that of Saturn, Jupiter will have gone quite round, and also as far in the next revolution as B_2, while Saturn has described part of a revolution only to C_2: then Jupiter will again have gone quite round, and also as far in the next revolution as B_3, while Saturn has described part of a revolution to C_3: then Jupiter will have performed a whole revolution, and part of another to B_1, while Saturn has performed part of a revolution to C_1: and then the same order of conjunctions will go on again. If, then, the periodic times were exactly in the proportion of 2 to 5, the conjunctions would continually take place in the same three points of the orbits. This conclusion will not be altered by supposing the orbits excentric: for though the places of conjunction may then be somewhat altered, the conjunctions, after the third, (when Saturn has gone round exactly twice, and Jupiter exactly five times,) will go on in the same order, and happen at the same places as before.

(165.) But the periodic times are not exactly in the proportion of 2 to 5, but much more nearly in the proportion of 29 : 72. This alters the distance of the places of conjunction. We must now suppose Saturn to move through 242°·79, and Jupiter (by the proportion just mentioned) will then have moved through 602°·79, or through a whole circumference and 242°·79, and they will be in conjunction again. The next conjunction will take place when Saturn has moved through double this angle, or 485°·58, or when Saturn has performed a whole revolution, and 125°·58 of the next revolution: and the following conjunction will take place when Saturn has moved through 728°·37, or when Saturn has gone twice round, and has described 8°·37 more. Now, then, the same order of conjunctions will not go on again at the same places as before, but the next three after this will be shifted

$8°·37$ before the former places, the three following the last-mentioned three will be again shifted $8°·37$, and so on. The places of successive conjunction, in *fig.* 38., will be at B_1, C_1, $b_2\,c_2$, $b_3\,c_3$, $b_4\,c_4$, $b_5\,c_5$, $b_6\,c_6$, &c. The shifting of the places of conjunction will take place in nearly the same manner, whether the orbits are excentric or not.

(166.) From this the following points are evident:—

First. In consequence of the periodic times being nearly in the proportion of 2 to 5, many successive conjunctions happen near to three equidistant points on the orbits.

Secondly. In consequence of the proportion being not exactly that of 2 : 5, but one of rather less inequality, the points of conjunction shift forward, so that each successive set of conjunctions is at points of the orbits more advanced, by $8°·37$, than the preceding one.

(167.) Let us now inquire how long it will be before the conjunctions happen at the same parts of the orbits as at first.

This will be when the series of points b_4, b_7, b_{10}, &c., extends to B_3. For then the series b_5, b_8, b_{11}, &c., will extend to B_1, and the series b_3, b_6, b_9, &c., will extend to B_2. The time necessary for this will be gathered from the consideration, that in three conjunctions the points are shifted 8°·37: and that the points must shift 120° from B_1, before they reach B_3: and that we may, therefore, use the proportion, As 8°·37 is to 3, so is 120° to 43 nearly, the number of conjunctions that must have passed before the points of conjunction are again the same. And as Saturn advances 242°·79 between any conjunction and the next, he will, at the forty-third conjunction from the first, have described 10440°, or 29 circumferences; and Jupiter, therefore, (by the proportion of their periodic times,) will have described 72 circumferences. The time, then, in which the conjunctions return to the same points is twenty-nine times Saturn's periodic time, or seventy-two times Jupiter's periodic time, or about 855 years *.

(168.) Now let us examine into the effects of this slow motion of the points of conjunction upon the forces which one body exerts to disturb the other.

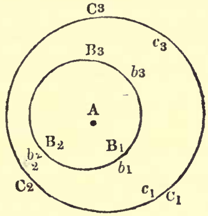
(169.) If the orbits had no independent excentricity, it would affect them no further than by the periodical distortion which would take place at

* These numbers are not quite exact: the proportion of 29 : 72 not being quite accurate.

every conjunction. There would be nothing in one set of conjunctions, more than in another, which could affect the dimensions of the orbits.

(170.) But if the orbits are not circular, this is no longer true. It is not the same thing whether the conjunctions take place at $B_1 C_1$, $B_2 C_2$, and $B_3 C_3$, *fig.* 39., or at $b_1 c_1$, $b_2 c_2$, and $b_3 c_3$. The distances of

Fig. 39.



the planets are not the same, and consequently the forces which they exert on each other are not the same; also their velocities are different in different parts of their orbits, or at different points of conjunction, and therefore the times during which they can act on each other are not the same. It is true that, in the figure, the distance at $b_2 c_2$ is less than at $B_2 C_2$, while that at $b_3 c_3$ is greater than at $B_3 C_3$; and thus there is a partial compensation in the changes of the effects produced in different points of the orbit. But it can be discovered only by very complete calculations, whether the compensation is perfect or not. The calculations necessary for this purpose are probably the most complicated that physical science has ever given occasion for; and the reader must not here expect the smallest account of them. This only can be stated as a result, that in no instance in the planetary system is the compensation perfect, and that the chances for its being perfect in any case are infinitely small.

(171.) We have here considered the varying influence of one body on the other at conjunction, as depending entirely on the excentricities of the two orbits. But there is another circumstance which may also cause the influence to vary. The orbits may be inclined, and this will affect both the distance of the bodies and the direction in which they attract each other.

(172.) In the case, then, of Jupiter and Saturn, we have the two planets acting on each other with forces which are nearly the same at every third conjunction, but are not exactly the same, and whose variations occupy a period of 850 years. Of these forces, parts are in the direction of the radius vector, and these tend directly to affect the major axes of the orbit described: other parts are perpendicular to the radius vector, sometimes accelerating and sometimes retarding; and these tend (though in opposite ways) to affect the major axes of the orbits. There are, therefore, forces tending to alter the major axes of the orbits, which go through all their changes only in 850 years. During half of this time they tend to make the major axis of Jupiter's orbit less, and that of Saturn's orbit greater; and during the other half they tend to make the major axis of Jupiter's orbit greater, and that of Saturn's orbit less. This coincidence, in time, of the increase of one major axis with the decrease of the other, is the result of investigation that we cannot explain here.

(173.) After the partial compensation that we have mentioned, it will readily be understood that the varying force which produces these effects is small. So small, indeed, is it, that after acting more than 400 years, it has increased (or diminished) the major axis of Saturn's orbit only by $\frac{1}{1350}$th part, and diminished (or increased) that of Jupiter's orbit only by $\frac{1}{8560}$th part. These alterations would hardly be discoverable with our best instruments. But during 400 years the major axis of each orbit differs from the major axis during the next 400 years by a part of these quantities: the planet's rate of annual angular motion is, for 400 years, constantly less than its average rate; and for the next 400 years it is constantly greater than its average rate: and in this length of time the inequality in longitude may (49.) grow up into a most formidable quantity. In fact, the inequality thus produced in Saturn's longitude amounts to about 48′, by which its true place is sometimes before and sometimes behind its mean place: that in Jupiter's longitude amounts to about 21′. (The greatest inequality of any other planet does not exceed 3′, and the greatest of the planets inferior to Jupiter does not exceed 25″.) The theoretical explanation of these inequalities was first given by Laplace in 1785.

(174.) The magnitude of these inequalities in the motions of Jupiter and Saturn, as we have seen, depends principally on the length of time during which the forces act in the same manner; first, because in this long time they can produce a sensible alteration in the major axis and annual angular motion; secondly, because the two planets move for so long a time with this altered angular motion. But it must also be borne in mind, that these two planets are by far the largest in the system; the mass of Jupiter being 300 times that of the earth, and the mass of Saturn being 100 times that of the earth (the next of the planets in the order of magnitude, except Uranus).

(175.) The same general reasoning, by which we have shown that there is a periodical inequality of the major axis of either of these orbits, will also show that there is a periodical inequality in the excentricity and in the place of the perihelion. It will also appear, in the same way, that these effects are the remainder, after partial compensation of effects in different parts of the orbit. Thus, if one conjunction happen when Jupiter is going toward aphelion, the effect of Saturn's disturbing force is to pull Jupiter from the sun; and therefore, by (59.), to increase the excentricity of Jupiter's orbit. But it is then perfectly certain that either the next conjunction, or the next but one, or perhaps both these, will happen at a part where Jupiter is going towards perihelion; and then, by (59.), the excentricity of Jupiter's orbit is diminished. Similar reasoning applies to the excentricity of Saturn's orbit. It becomes, then, a matter of calculation, whether the compensation is perfect or not. Now it appears, upon investigation, that the compensation is not perfect, but that, while the points of conjunction shift through 120°, the effect of the uncompensated part is, for half the time, to increase the excentricity, and for half the time to diminish it. It appears, also, that there is no necessary connexion between the time at which the excentricity is greatest or least, and that when the major axis is greatest or least; so that we cannot assert that when the major axis is greatest the excentricity is greatest, or the contrary; or that the excentricity of one is greatest when that of the other is greatest: all that we can assert is, that the excentricity of each orbit occupies the same time in going through its changes from greatest to least, as the major axis occupies in going through its change from greatest to least. The effect on the planet's distance from the sun, produced by the change of excentricity, is much more considerable than that from the change in the major axis; being for Jupiter $\frac{1}{1230}$ of his whole distance, and for Saturn $\frac{1}{314}$ of his whole distance.

(176.) Similar remarks apply, in every respect, to the motion of the perihelion of each orbit. Each is made to progress during 425 years, and to regress during 425 years; but there is no necessary relation between the time when one has progressed furthest, and the time when the other has progressed furthest. There is, however, a necessary relation between the change of excentricity and the motion of the perihelion of each orbit: the excentricity of either orbit has its mean value when the perihelion of that orbit has progressed furthest or regressed furthest; and when the excentricity is either greatest or least, the perihelion is at its mean place.

(177.) We have taken the long inequality of Jupiter and Saturn as the most imposing by its magnitude, and the most celebrated for its history (as, before it was explained theoretically, astronomers were completely bewildered by the strange irregularity in the motion of these planets). But there are several others which, in theory, are as curious. Eight times the periodic time of the earth is very nearly equal to thirteen times the periodic time of Venus; and this produces, in the motions of the earth and Venus, a small inequality, which goes through all its changes in 239 years. Four times the periodic time of Mercury is nearly equal to the periodic time of the earth, and this produces an inequality whose period is nearly 7 years. The periodic time of Mars is nearly double of the earth's, and this produces a considerable inequality, depending on the excentricities, &c., besides that mentioned in (159.), which was independent of the excentricities. Twice the periodic time of Venus is nearly equal to five times that of Mercury; three times the periodic time of Venus is nearly equal to that of Mars; three times the periodic time of

Saturn does not much differ from that of Uranus. Each of these approximations to equality gives rise to an equation of sensible magnitude, and of long period, in the motion of both planets.

(178.) But it will easily be seen that the defect of compensation, on which the effects depend, is much greater in some cases than in others. The conjunctions of the earth and Mars take place at only one point, and the points near it, for several revolutions: those of Venus and Mars take place only at two opposite points and their neighbourhood, (as each successive conjunction takes place when Mars has described half a revolution, and Venus $1\frac{1}{2}$ revolution;) those of Jupiter and Saturn, as we have seen, at three points; those of Venus and the earth at five points. It is evident that, in the first of these, the whole effect of the change of one point of conjunction has its influence in altering the orbit's dimensions; that in the second there is only the difference between two effects; that in the third there is the mixture of three, which tend to balance; that in the next there is the mixture of five in the same way. The smaller, then, is this number of points, the more favourable are the circumstances (supposing the same length of period for the inequality) for producing a large inequality. This number of points is always the same as the difference between the two least numbers, expressing nearly the proportion of the periodic times. Thus we may expect to find a large inequality when the periodic times of two planets are very nearly in the same proportion as two numbers, whose difference is small.

(179.) We shall now proceed to mention the *secular* variations of the elements of the orbits of planets. By this term is meant those variations which do not depend upon the positions of the planets in their orbits, or the places of conjunction, but merely upon their relative distances and excentricities, and the positions of their lines of apses. They are, therefore, the variations which depend upon the mean or average action of one planet upon another in the long run: all the sensible departures from the secular variation, produced by the irregularity of the action of one planet upon another, being supposed to be contained in the inequalities already discussed.

(180.) First, then, with regard to the mean distance of a planet. If we consider an exterior planet disturbing an interior one, (as Saturn disturbing Jupiter,) the disturbing force in the direction of the radius vector, by (77.), &c., tends sometimes to draw it from the sun, sometimes to draw it towards the sun, but the former is the greater, and we may therefore consider the force as, upon the whole, diminishing the sun's attraction. This, by (46.), alters the relation between the periodic time and the mean distance, so that the mean distance is less than it would have been with the same periodic time, had there been no disturbance. If we consider an interior planet disturbing an exterior one, (as Jupiter disturbing Saturn,) the disturbing force tending to draw it to the sun is greatest; and here the mean distance is greater than it would have been with the same periodic time, had there been no disturbance. But so long as these general effects in the force directed to the sun continue unaltered, the mean distances will not alter (46.), &c. Now, upon taking a very long period, (as several thousand years,) it is easy to see that, if we divide that period into two or three parts, the two planets have in each of those parts been in conjunction indifferently in all parts of their orbits; that they have had every possible relative position in every part; and that (if we make the periods long enough) the force which one planet has sustained in any one point will be accurately the mean of all which it would sustain, if we estimated all those that it could suffer from supposing the other planet to go with its usual motion through the whole of its orbit. As this mean will be the same for each of the periods, there will, in the long run, be no alteration of the force in the direction of the radius vector, and we may assert at once that the mean distance cannot be altered by it.

(181.) But with regard to the disturbing force acting perpendicularly to the radius vector, the circumstances are different. The mere existence of such a force, without variation, causes an alteration in the mean distance (48.); and it is necessary to show that the nature and variations of the force are such that, in the long run, the velocity of the disturbed planet is not affected by it. For this purpose, instead of considering merely the disturbing force perpendicular to the radius vector, we will consider separately the whole force which the disturbing planet exerts on the sun, and the whole force which it exerts on the disturbed planet. Now, the force which it exerts on the sun tends to pull the sun sometimes in one direction and sometimes in another, but, on the whole, produces no permanent displacement: this force, then, may at once be neglected. The force which one planet has exerted on the other has acted when, for any arbitrary position of the disturbing planet, the disturbed planet has been at every point of its orbit. Since the whole acceleration produced in a long time is the sum of all the accelerations diminished by the sum of all the retardations, we may divide them into groups as we please, and sum each group. Let us, then, group together all the accelerations and retardations produced in one position of the disturbing planet. The disturbed planet having been in every small part of its orbit, during a time proportional to the time which it would occupy in passing through that small part in any one revolution, the various accelerations and retardations will bear the same proportion as if the disturbed planet had made one complete revolution, and the disturbing planet had been fixed. Now, it is a well-known theorem of mechanics, that when a body moves through any curve, acted on by the attractions of any fixed bodies, its velocity, when it reaches the point from which it started, is precisely the same as when it started: the accelerations and retardations having exactly balanced. Consequently, in the case before us, if the disturbing planet had been fixed, and the disturbed planet had made one complete revolution, the latter would, on the whole, have been neither accelerated nor retarded; and, therefore, in the long run, all the accelerations and retardations of the disturbed planet, produced in any arbitrary position of the disturbing planet, will exactly balance. The same may be shown for every position of the disturbing planet; and thus, on the whole, there is no alteration of velocity. Since, then, in the long run, the planet's velocity is not altered, and since (180.) the force directed to the sun is not altered, the planet's mean distance will not be altered. This reasoning does not prevent the increase or diminution of the velocity at particular parts of the orbit, and therefore the eccentricity and the line of apses may vary; but it shows that, if there is an increase at one part, there is a diminution that balances it at another; and at the point where the orbit at the beginning of a long time, and the orbit at the end of that time intersect, (which will be at mean distance nearly,) the velocity will not be altered.

Our demonstration supposes that the portions of the curves described in different revolutions, for the same position of the disturbing planet, are parts of one orbit, and therefore does not take account of the alteration in the magnitude of the disturbing force produced by the alteration of place which that force has previously caused. This has been taken into account, to a certain degree, by several mathematicians, and it appears, as far as they have gone, that this produces no alteration in the conclusion.

(182.) Secondly, as to the place of perihelion, or the position of the line of apses. The motion of this will depend essentially on the excentricity of the orbit of the disturbing planet. Suppose, for instance, that the orbit of Venus was elliptical and the earth's orbit circular; as the distance of these planets in conjunction is little more than $\frac{1}{4}$th of the earth's distance from the sun, the ellipticity of the orbit of Venus would bring that planet at aphelion so much nearer to the earth's orbit, that by far the greatest effect would take place when in conjunction there; and this, by (54.), would make Venus' line of apses progress. But if the earth's orbit were more elliptic than that of Venus, and if the earth's perihelion were on the same side of the sun as the perihelion of Venus, it might happen that the principal action would take place at perihelion, and then, by (51.), the line of apses would regress. These effects would continue to go on, while the relative position of the lines of apses, and the proportion of the excentricities, remained nearly the same. As, in the long run, conjunctions would happen everywhere, the preponderating effect would be similar to the greatest effect; and thus, the secular motion of the line of apses will be constant, (till the positions of the lines of apses, &c. shall have changed considerably;) its magnitude and direction will depend on the excentricities of both orbits; but if the disturbed planet is the interior, and if the orbit of the other be not excentric, the line of apses will progress. The same is true, if the disturbed planet is exterior (the greatest action being then at the perihelion, if the interior orbit have no excentricity, and being directed to the sun.)

(183.) Thirdly, as to the excentricity. If the orbit of the disturbing planet were circular, the effect on the excentricity produced by conjunction at the place where the orbits are nearest, would be of one kind before conjunction, and of the opposite kind after conjunction, from the disturbing force in the radius vector, as well as from that perpendicular to the radius vector; and thus the excentricity would not be altered. The same would happen if both orbits were excentric, provided their lines of apses coincided. Thus it appears that there is no variation of excentricity, except the orbit of the disturbing planet is excentric, and its line of apses does not coincide with that of the disturbed planet. When these conditions hold, (as they do in every planetary orbit,) a general idea of the effect may be obtained by finding where the orbits approach nearest; then, if we consider the disturbance of the interior planet, since the force draws it from the sun, the excentricity will be increased if it is moving from perihelion, or diminished if it is moving towards perihelion. For the exterior planet, as the force draws it towards the sun, the conclusion will be of the opposite kind. These effects are constant, till the excentricities and the positions of the lines of apses have changed sensibly. The place where the force at conjunction produces the greatest effect on the excentricity may not be strictly the place where the orbits are nearest, but probably will not be far removed from that place.

At the place where the orbits approach nearest, both planets in general are moving from perihelion, or both towards perihelion, so that when one excentricity is increased, the other is diminished.

(184.) For the general stability of the planetary system, the positions of the lines of apses are not important, but the permanency of the major axes and the excentricities are of the greatest importance. The conclusion which we have mentioned as to the absence of secular variation of the major axis, from the action of one planet, applies also to the disturbances produced by any number of planets, and thus we can assert that the major axes of the orbits of the planets are not subject to any secular variation. The excentricities are subject to secular variation, but even this corrects itself in a very long time: when the investigation is fully pursued, it is found that each of the excentricities is expressed by a number of periodic terms, the period of each being many thousands of years. Thus the major axis of the earth's orbit, notwithstanding its small and frequent variations, has not sensibly altered in many thousands of years, and will not sensibly alter; the excentricity, besides suffering many small variations, has steadily diminished for many thousands of years, and will diminish for thousands of years longer, after which it will again increase.

(185.) A remarkable relation exists between the variation of the excentricities, (of which that mentioned in (183.) is a simple instance,) the result of which, as to the state of the excentricities at any time, is given thus:—The sum of the products of the square of each excentricity by the mass of the planet, and by the square root of the major axis, is always the same.

Section VIII.—*Perturbation of Inclination and Place of Node.*

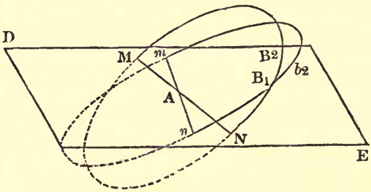
(186.) We have hitherto proceeded as if the sun, the moon, and all the planets, revolved in the same plane—as if, for instance, the sun were fixed in the centre of a table, and all the planets, with their satellites, revolved on the surface of the table. But this supposition is not true. If we suppose the earth to revolve on the surface of the table, the moon will, in half her revolution, (we mean while she describes 180°, not necessarily in half her periodic time,) rise above the table, and in the other half she will go below it, crossing the surface at two points which, as seen from the earth, are exactly opposite. Venus will, in half her revolution, rise above the table, and in half will sink below it, crossing the table at two points which, as seen from the sun, are exactly opposite; each of the other planets and satellites in like manner crosses the plane at points which, as seen from the central body, are exactly opposite. In different investigations it is necessary to consider the inclination of the plane of revolution or the plane of the orbit to different planes of reference: the line in which the plane of revolution crosses the plane of reference is called the *line of nodes on that plane;* and the angle which the plane of revolution makes with the plane of reference is called the *inclination to that plane.* The plane of reference must always be supposed to pass through the central body.

(187.) The inclinations of all the orbits, except those of the small planets, are so trifling, (the largest—namely, that of the moon's orbit to the earth's orbit—being, at its mean state, only 5°,) that they may in general be wholly neglected in estimating the disturbance which one planet produces in the motion of another in its own plane. In some cases, however, as in the inequalities of long period, where the effective force is only the small part which remains after a compensation more or less perfect, no alteration of the forces must be neglected; and here, as we have hinted in (171.), the inclinations must be taken into account.

(188.) But though the alteration which the inclination produces in the forces that tend to disturb the body's motion *in* its plane may, in most cases, be neglected, yet the force which tends to pull the body *above* the plane or *below* the plane cannot be neglected. In almost every case this force will be less than the force tending to disturb the motion in the plane, yet it will be much greater than the alteration which the inclination produces in that force. It is our object in this section to show the nature of the alteration which is produced by the force tending to pull the body from the plane.

(189.) First, then, as to the effect of a force generally which acts perpendicularly to the plane of revolution. (We shall confine ourselves at present to forces which act perpendicularly to the plane, because it is evident that forces which act in, or parallel to, the plane of the orbit, whether in the radius vector or perpendicularly to it, will not cause the planet to depart from that plane.) Let *fig.* 40, be a perspective representation of an orbit, and a plane of reference. Suppose M A N to be

Fig. 40.

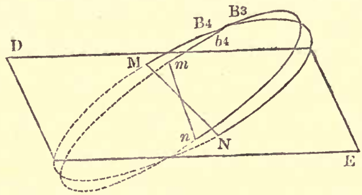


the line of nodes at which the plane of the orbit $N B_1 B_2$ crosses the plane of reference D E; the central body A being in the line of nodes, and the part of the orbit marked by a dark line being above the plane, and that marked by a dotted line being below it. Suppose that the planet has moved from N to B_1, and that at B_1, before it reached the point highest above the plane D E, a force pulls it down towards the plane. After a short time, instead of going to B_2, where it would have been if no force had disturbed it, it will be found at b_2, having described $B_1\,b_2$, instead of $B_1\,B_2$. It is plain that the orbit in which the planet must have moved without a disturbing force, in order to describe $B_1 b_2$ now, could not be N B^1, but must be such a curve as n B_1, crossing the plane D E at a point in the situation of the point n. Therefore, if no more disturbing forces act, the planet, which has described B_1 b_2 as if it came without disturbance from n, will go on to describe an orbit as if it had come without disturbance from n, and will therefore describe an orbit n B_1 $b_2\,m$, crossing the plane E F in the points n and m. The line of nodes is changed from M A N to m A n.

(190.) Here the line of nodes has twisted in a direction opposite to the planet's motion, or has *regressed*. The inclination of the new plane is evidently less than that of the old one, since it passes through the same point B, and cuts the plane of reference in a line more distant from B than the line in which the old one cut it, or the inclination is *diminished*.

(191.) Now, if we conceive that at B_3, *fig.* 41, after the planet has passed the point highest above the plane, a force tends to pull it towards the plane, the planet, instead of going to B_4, will go to b_4, and

Fig. 41.



instead of crossing the plane D E at M, will cross it at *m*; and then, if it is not disturbed again, will proceed in an orbit of which B_3 b_4 *m* is a part, and which will cross the plane D E at the points *m* and *n*. The new line of nodes has twisted here also in the direction opposite to the direction of the planet's motion, or has *regressed.* But the inclination of the new plane is greater than that of the old one, since it passes through the same point B_3, and cuts the plane of reference in a line less distant from B_3 than the line in which the old one cut it, or the inclination is *increased.*

(192.) We have, then, this general result: If a force acting perpendicularly to the orbit tends to draw the planet towards the plane of reference, it always causes the line of nodes on that plane to regress: while the planet is moving from a node to the point highest above the plane of reference, it diminishes the inclination to that plane; and while the planet is moving from the highest point to a node, it increases the inclination.

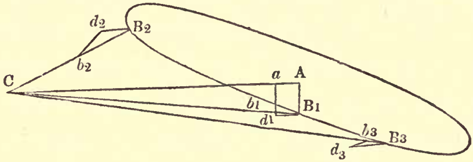
(193.) In the same manner, if the force tends to draw the planet from the plane of reference, it always causes the line of nodes to progress. While the planet is moving from a node to the point highest above the plane, it increases the inclination; and while the planet is moving from the highest point to a node, it diminishes the inclination.

(194.) Similar results would have been obtained if we had considered the action of the force while the planet is in that part of its orbit which is on the other side of the plane DE.

We shall now proceed with the consideration of the force perpendicular to the orbit, which is produced by the attraction of a disturbing body.

(195.) First: it is plain that, if the disturbing body is in the plane of the orbit (produced, if necessary), it will not tend to draw either the central body or the planet out of that plane, and therefore will produce no disturbing force perpendicular to the plane of the orbit. Proceeding, then, with the supposition that the disturbing body is not in the plane of the orbit; and supposing *fig.* 42, to be a perspective view of an orbit $B_2 B_1 B_3$ (which, to assist our ideas, may be conceived to differ little from a circle) with the disturbing body C out of the plane of the orbit, let us take three points B_1 B_2 B_3, of

Fig. 42.



which B_1 is at the same distance as A from C, B_2 is nearer to C, and B_3 farther from C than A is. Suppose that the attraction of C draws A in a certain small time through the space A a, and that when the planet is at B_1, or B_2, or B_3, the attraction draws the planet in the same time through $B_1 b_1$, or B_2 b_2, or $B_3 b_3$ respectively. Then (as in (71.)) the attraction of C upon the two bodies A and B would produce no disturbance in their relative motions, if it drew them through equal spaces in the same direction. Draw B_1 d_1, B_2 d_2, B_3 d_3 each equal and parallel to Aa; then if the attraction had drawn B_1 to d_1, there would have been no disturbance, and consequently the real disturbance at B_1 is represented by a force which would have drawn the planet from d_1 to b_1. Similarly, the real disturbances at B_2 and B_3 are represented by forces which would have drawn the planet from d_2 to b_2, and from d_3 to b_3 respectively. Now, since C B_1 is equal to C A, the forces of C upon A and B_1 are equal, and therefore $B_1\, b_1$ is equal to A a, and therefore $a\, b_1$ is parallel to A B_1, and therefore is in the same straight line with $b_1\, d_1$; and consequently at B_1 the whole disturbing force is parallel to the radius vector, and there is no part perpendicular to the plane of the orbit. But at B_2 the planet is nearer to C, the force therefore on the planet is greater, and $B_2\, b_2$ is therefore greater than A a or $B_2\, d_2$; also it is more nearly perpendicular to the plane of the orbit than $B_2\, d_2$; and consequently b_2 is farther from the plane of the orbit than d_2; and therefore the disturbing force $d_2\, b_2$ is directed from the plane of the orbit towards the side on which C is. On the contrary, at B_3 the planet is farther from C; the force on the planet is therefore less: and $B_3\, b_3$ is therefore less than A a or $B_3\, d_3$; moreover it is inclined more to the perpendicular than $B_3\, d_3$, and consequently b_3 is nearer to the plane of the orbit than d_3; and therefore the disturbing force $d_3\, b_3$ is directed from the side on which C is. Thus we find,

(196.) When the central and revolving bodies are equally distant from the disturbing body, there is no disturbing force perpendicular to the plane of the orbit.

(197.) When the revolving body is nearer the disturbing body than the central body is, the disturbing force perpendicular to the plane tends to draw the revolving body out of the plane to that side on which the disturbing body is.

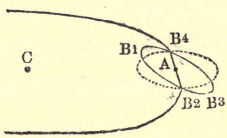
(198.) When the revolving body is farther from the disturbing body than the central body is, the disturbing force perpendicular to the plane tends to draw the revolving body out of the plane to the side opposite the disturbing body.

We may now apply these conclusions to the alteration of the node and inclination of the moon's orbit produced by the sun's attraction. The plane of reference is here supposed to be the plane of the earth's orbit.

(199.) First: suppose the line of nodes of the moon's orbit to be in syzygies, or to pass through the sun. Here the sun is in the moon's orbit produced, and therefore by (189), there is no disturbing force perpendicular to the moon's orbit.

(200.) Secondly: suppose the line of nodes to be in quadratures, or to be perpendicular to the

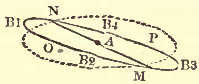
Fig. 43.



line drawn from the earth to the sun, as in *fig.* 43. The sun, in the figure, may be considered as being below the plane of the moon's orbit. Also, the moon's distance from the earth being small, the points, at which the moon's distance from the sun is the same as the earth's, are very nearly the same as the points of quadrature, or (in the case before us) they are very nearly the same as the nodes. Consequently, while the moon moves from B_4 through B_1 to B_2, she is nearer to the sun than the earth is, and therefore the disturbing force, by (197.), tends to pull her downwards from the plane of her orbit: while the moon moves from B_2, through B_3, to B_4, she is farther from the sun than the earth is, and therefore the disturbing force tends to pull her upwards from the plane of her orbit. In the case before us, then, the disturbing force is always directed towards the plane of reference. Consequently, by (192.), while the moon moves from B_4 to B_1, the line of nodes is made to regress, and the inclination is diminished; while the moon moves from B_1 to B_2, the line of nodes regresses, and the inclination is increased; while the moon moves from B_2 to B_3, the line of nodes regresses, and the inclination is diminished: and while the moon moves from B_3 to B_4, the line of nodes regresses, and the inclination is increased. The inclination, therefore, is not sensibly altered in a whole revolution, but the line of nodes regresses during the whole of the revolution.

(201.) Thirdly: suppose the line of nodes to be in such a position that the moon passes the line of nodes in going from quadrature to syzygy, as in *fig.* 44. Here the sun is to be considered as below the moon's orbit, and, therefore, while the moon moves from B_4, through B_1 to B_2, the

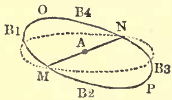
Fig. 44.



disturbing force tends to pull her down from the plane of the orbit, and while she moves from B_2, through B_3 to B_4, the force tends to pull her up from the plane of her orbit. Therefore, in going from B_4 to N, the force pulls the moon from the plane of reference; and causes thereby a progression of the line of nodes and a diminution of the inclination (193.); in going from N to the highest point O, the force pulls the moon towards the plane of reference; and, therefore, causes the nodes to regress, and the inclination to diminish, (192.); in going from the highest point O to B_2, the force still pulls the moon towards the plane of reference; and, therefore, still causes the nodes to regress, but causes the inclination to increase. Thus while the moon moves from B_4 to N, the force causes the line of nodes to progress, and while she moves from N to B_2, it causes the line of nodes to regress; and, similarly, while she moves from B_2 to M, the force causes the line of nodes to progress; and while she moves from M to B_4, it causes the line of nodes to regress. On the whole, therefore, the line of nodes regresses, but not so rapidly as in the second case. Also, while the moon moves from B_4 to O, the inclination is diminished, and while she moves from O to B_2 the inclination is increased; and, similarly, while she moves from B_2 to P the inclination is diminished; and while she moves from P to B_4 the inclination is increased. On the whole, therefore, the inclination is diminished.

(202.) Fourthly: suppose the line of nodes to be in such a position that the moon passes it in going from syzygy to quadrature, as in *fig.* 45.

Fig. 45.



Here, also, the sun is below the plane of the orbit produced; and, therefore, from B_4 to B_2 the force tends to pull the moon down from her orbit; and from B_2 to B_4 it tends to pull her up from it. As in the last case it would be seen, that while the moon moves from B_4 to M, the line of nodes regresses, while from M to B_2, the line of nodes progresses; while from B_2 to N, the lines of nodes regresses; and while from N to B_4, the line of nodes progresses. On the whole, therefore, the line of nodes regresses. Also, it will be seen, that while the moon moves from B_4 to O, the inclination is diminished; while from O to B_2 the inclination is increased; while from B_2 to P, the inclination is diminished; and while from P to B_4, the inclination is increased. On the whole, therefore, the inclination is increased.

The same reasoning would apply, and lead to the same conclusions in every respect, if we supposed the moon's orbit inclined in the opposite direction.

(203.) Now the earth moves round the sun, and, therefore, the sun appears to move round the earth, in the same direction in which the moon moves round the earth. If then we begin with the state in which the line of nodes is passing through the sun (and in which neither the node nor the inclination undergoes any change, by the first case), we come next to the state in which the moon passes the line of nodes in going from quadrature to syzygy (in which the node regresses and the inclination diminishes, by the third case); then we come to the state in which the line of nodes coincides with the line of quadratures (in which the node regresses rapidly, and the inclination is not altered, by the second case); then we come to the state in which the moon passes the line of nodes in going from syzygy to quadrature (in which the node regresses and the inclination is increased, by the fourth case); and then we come to the state in which the line of nodes again passes through the sun. This is when the sun has described, apparently, half a revolution round the earth (or rather less, in consequence of the regression of the node), and in the other half revolution, the same changes in every respect take place in the same order. The inclination, therefore, is greatest when the line of nodes passes through the sun, or coincides with the line of syzygy; and is least when the line of nodes coincides with the line of quadratures; since it is constantly diminishing while we are going from the former state to the latter, and constantly increasing while we are going from the latter state to the former. This is the principal irregularity in the inclination of the moon's orbit; all the others are very small.

(204.) The line of nodes is constantly regressing at every revolution of the moon, except when the line of nodes passes through the sun. The annual motion which we might at first expect it to have, is somewhat diminished by the circumstance, that the rapid regression of the line of nodes, when in the position in which the greatest effect is produced, carries it from the line of quadratures more swiftly than the sun's progressive motion only, by making the line of quadratures to progress, would separate them. But as the line of nodes never progresses, the diminution of the motion of the line of nodes occasioned thus, is very much less than the increase of the motion of the line of apses, (107.) Also, as the force acting on opposite points of the orbit, tends to produce effects of the same kind, there is no irregularity similar to that explained in (106.) Hence the actual regression of the line of nodes, though a little less than might at first be expected, differs from that regression by a much smaller quantity than that, by which the actual motion of the line of apses differs from the motion which at first we might expect it to have. The line of nodes revolves completely round in something more than nineteen years.

(205.) The effect of the irregularity in the regression of the nodes, and the effect of the alternate increase and diminution of the inclination, are blended into one inequality of latitude, which depends on the sun's longitude, the longitude of the moon's node, and the moon's longitude. This inequality was discovered (from observation) by Tycho Brahe, about A.D. 1590. It may be considered to bear the same relation to the inclination which the evection bears to the excentricity; and, like the evection in longitude, it is the greatest of the inequalities in latitude. It is, however, much less than the evection; its greatest effect on the moon's latitude being about 8′, by which the mean inclination is sometimes increased and sometimes diminished.

(206.) There are other small inequalities in the moon's latitude, arising partly from the changes in the node and inclination, which take place several times in the course of each revolution, (200.), &c.; partly from the excentricity of the orbits of the moon and the earth, partly from the distortion accompanying the *variation*, and partly from the variability of the inclination itself. We shall not, however, delay ourselves with the explanation of all these terms.

(207.) We shall now proceed with the disturbance of the planets in latitude.

In this inquiry it is always best to take the orbit of the disturbing planet for the plane of reference. Now let us first consider the case of Mercury or Venus disturbed by Jupiter. In this case, Jupiter revolving in a long time round the sun, which is the central body to Mercury or Venus, produces exactly the same effect as the sun revolving (or appearing to revolve) round the earth, which is the central body to the moon. The disturbing force of Jupiter, therefore, produces a regression of the nodes of the orbits of Mercury and Venus on Jupiter's orbit; and an irregularity in the motion of each node, and an alteration in the inclination, whose effects might be combined into one: and this is the only inequality in their latitude, produced by Jupiter, whose effects are sensible.

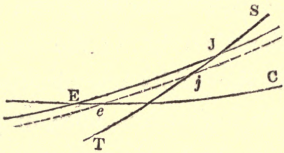
(208.) The other inequalities in latitude, depending on the relative position of the planets, possess no particular interest; and a general notion of them may be formed from the remarks in the discussion of the motion of the moon's node. One case, however, may be easily understood. When an exterior planet is disturbed by the attraction of an interior planet, whose distance from the sun is less than half the distance of the exterior planet, and whose periodic time is much shorter, then the exterior planet is always further from the interior planet than the sun is, and therefore, by (195.), there is a disturbing force urging it from the plane of reference when the planets are in conjunction, and to it when they are in opposition; and thus the exterior planet is pushed up and down for every conjunction of the two planets. The disturbance, however, is nothing when the exterior planet is at the line of its nodes (195.).

(209.) The near commensurability of periodic times, which so strikingly affects the major axis, the excentricity, and the place of perihelion, produces also considerable effects on the node and the inclination. The reasoning of (175.) and (176.) will in every respect apply to this case: the greatest effect is produced, both on the motion of the node and on the change of inclination, when the planets are in conjunction: the gradual alteration of the point of conjunction produces a gradual alteration of these effects, which, however (in such a case as that of Jupiter and Saturn), is partially counteracted by the gradual change of the other points of conjunction: the uncompensated part, however, may, in many years, produce a very sensible irregularity in the elements. If we put the words *line of nodes* for *line of apses,* and *inclination* for *excentricity,* the whole of the reasoning in (175.), &c., will apply almost without alteration.

(210.) For the secular variation of the position of the orbit, the following considerations seem sufficient. In the long run, the disturbed planet has been at every one point of its orbit a great number of times, while the disturbing planet has been at almost every part of its orbit. The disturbing force is always the difference of the forces which act on the sun and on the disturbed planet. As the disturbing planet, in these various positions, acts upon the sun in all directions in the plane of its orbit, its effect on the sun may be wholly neglected; and then it is easy to see that, whether the disturbing planet be exterior or interior to the other, the combined effect of the forces in all these points on the disturbed planet at one point, is to pull it from its orbit towards the plane of the disturbing planet's orbit. (This depends upon the circumstance that the force is greatest when the disturbing planet is nearest.) Consequently, by (192.), the line of nodes of the disturbed planet's orbit on the disturbing planet's orbit, in the long run, always regresses. If the orbits are circular, there is no alteration of the inclination, since, at points equally distant from the highest point, there is the same force on the disturbed planet; and, therefore, by (192.), the inclination is increased at one time, and diminished as much at another. If the orbits are elliptic, one point may be found where the effect of the force on the inclination is greater than at any other, and the whole effect on the inclination will be similar to that.

(211.) In stating that the nodes always regress in the long run, the reader must be careful to restrict this expression to the sense of regressing on the orbit of the disturbing planet. It may happen that on another orbit they will appear to progress. Thus the nodes of Jupiter's orbit are made to regress on Saturn's orbit by Saturn's disturbing force. The nodes of these orbits on the earth's orbit are not very widely separated: but the inclination of Saturn's orbit is greater than that of Jupiter's. If we trace these on a celestial globe, we shall have such a figure as *fig.* 46.; where E C represents the plane of the earth's orbit, J E the orbit of Jupiter, and S T that of Saturn. The orbit of Jupiter, by

Fig. 46.



regressing on Saturn's orbit, assumes the position of the dotted line *j e;* but it is plain that the intersection of this orbit with the earth's orbit has gone in the opposite direction, or has progressed. If the motion of the node on Saturn's orbit from J to *j* is regression, the motion of the node on the earth's orbit from E to *e* must be progression.

(212.) There is a remarkable relation between the inclinations of all the orbits of the planetary system to a fixed plane, existing through all their secular variations, similar to that between their excentricities. The sum of the products of each mass, by the square root of the major axis of its orbit, and by the square of the inclination to a fixed plane, is invariable.

(213.) The disturbance of Jupiter's satellites in latitude presents circumstances not less worthy of remark than the disturbance in longitude. The masses are so small, and their orbits so little inclined to each other, that the small inequalities produced in a revolution may be neglected. Even that depending on the slow revolution of the line of conjunctions of the first three satellites, so small is the mutual inclination of their orbits, does not amount to a sensible quantity. We shall, therefore, consider only those alterations in the position of the planes of the orbits which do not vary sensibly in a small number of revolutions. For this purpose, we must introduce a term which has not been introduced before.

(214.) If the moon revolved round the earth in the same plane in which the earth revolves round the sun, the sun's attraction would never tend to draw the moon out of that plane. But (taking the circumstances as they really exist,) the moon revolves round the earth in a plane inclined to the plane in which the earth revolves round the sun; and the consequence, as we have seen, is, that the line of nodes upon the latter plane regresses, and the inclination of the orbit to the latter plane remains, on the whole, unaltered. The plane of the earth's orbit, then, may be considered a *fundamental plane* to the moon's motion; by which term we mean to express, that if the moon moved in that plane, the disturbing force would never draw her out of it; and that if she moved in an orbit inclined to it, the orbit would always be inclined at nearly the same angle to that plane, though its line of nodes had sensibly altered. The latter condition will, in general, be a consequence of the former.

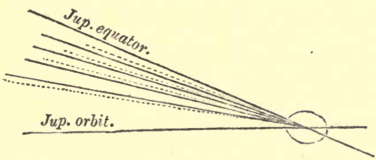
(215.) In order to discover what will be the fundamental plane for one of Jupiter's satellites, we must consider that, besides the sun's attraction, there is another and more powerful disturbing force acting on these bodies, namely, the irregularity of attraction produced by Jupiter's flatness. The effect of this (as we shall show) is always to pull the satellites towards the plane of Jupiter's equator. If Jupiter were spherical, the only disturbing force would be the sun's attraction, tending on the whole to draw the satellite towards the plane of Jupiter's orbit, and that plane would be the fundamental plane of the satellite. If Jupiter were flattened, and if the sun did not disturb the satellite, the irregularity in Jupiter's shape would always tend to draw the satellite towards the plane of his equator, and the plane of his equator would be the fundamental plane of the satellite. As both causes exist, the position of the actual fundamental plane must be found by the following consideration. We must discover the position of a plane from which the sun's disturbing force tends, on the whole, to draw the satellite downwards, and the disturbing force depending upon Jupiter's shape, tends to draw it upwards (or *vice versâ*) by equal quantities; and that plane will be the fundamental plane. This plane must lie *between* the planes of Jupiter's orbit and Jupiter's equator, because thus only can the disturbing forces act in opposite ways, and therefore balance each other: and it must pass through their intersection, as otherwise it would at that part be above both or below both, and the forces depending on both causes would act the same way.

(216.) The disturbing force of the sun, as we have seen, (82.), &c., is greater as the satellite is more distant; the disturbing force depending on Jupiter's shape is then less, as we shall mention hereafter. Consequently, as the satellite is more distant, the effect of the sun's disturbing force is much greater in proportion to that depending on Jupiter's shape. Thus, if there were a single satellite at the distance of Jupiter's first satellite, its fundamental plane would nearly coincide with the plane of Jupiter's equator; if, at the distance of Jupiter's second satellite, its fundamental plane would depart a little farther from coincidence with the plane of the equator; and so on for other distances; and if the distance were very great, it would nearly coincide with the plane of Jupiter's orbit. If, then, Jupiter's four satellites did not disturb each other, each of them would have a separate fundamental plane, and the positions of these planes would depend only upon each satellite's distance from Jupiter.

(217.) In fact, the satellites do disturb each other. In speaking of the planets (210.), we have observed that the effect of the attraction of one planet upon another, in the long run, is to exert a disturbing force tending to draw that other planet (at any part of its orbit) towards the plane of the first planet's orbit. The same thing is true of Jupiter's satellites. Now, though each of them moves generally in an orbit inclined to its fundamental plane, yet in the long run (when the nodes of the orbit have regressed many times round,) we may consider the motion of each satellite as taking place in its fundamental plane. The question, therefore, must now be stated thus. The four satellites are revolving in four different fundamental planes; and the position of each of these planes is to be determined by the consideration that the satellite in that plane is drawn towards the plane of Jupiter's orbit by the sun's disturbing force, towards the plane of Jupiter's equator by the force depending on Jupiter's shape, and towards the plane of each of the other three satellites, by the disturbing force produced by each satellite: and these forces must balance in the long run.

(218.) The determination of these planes is not very difficult, when general algebraical expressions have been investigated for the magnitude of each of the forces. The general nature of the results will be easily seen; the several fundamental planes will be drawn nearer together (that of the first satellite, that of the second, and that of the third, will be drawn nearer to Jupiter's orbit, while that of the fourth will be drawn nearer to Jupiter's equator.) The four planes will still pass through the intersection of the plane of Jupiter's equator with that of Jupiter's orbit. Thus, if we conceive the eye to be placed at a great distance, in the intersection of the planes of Jupiter's orbit and Jupiter's equator, and if the dotted lines in *fig.* 47 re-

Fig. 47.



present the appearance of the fundamental planes which would exist if the satellites did not disturb each other, then the dark lines will represent the positions of these planes as affected by the mutual disturbances. The inclination of Jupiter's equator to Jupiter's orbit is about 3° 5′; and so great is the effect of his shape, that the fundamental plane of the first satellite is inclined to his equator by only 7″; that of the second satellite by about 1′; that of the third by about 5′; and that of the fourth by about 24′½. Without mutual perturbation, the inclinations to Jupiter's equator would have been about 2″, 20″, 4′, and 48′.

(219.) Having considered the positions of the fundamental planes, we shall now consider the motion of a satellite, when moving in an orbit inclined to its fundamental plane.

(220.) The general effect will be of the same kind as that for the moon. Since the disturbing force which then tends to pull it from the plane of its orbit, tends to pull it towards the fundamental plane (as, supposing the satellite to be on that side of the fundamental plane next the plane of Jupiter's equator, the sun's disturbing force towards Jupiter's orbit is increased, that towards Jupiter's equator is diminished, and so for the others), the line of nodes will regress on the fundamental plane. The inclination on the whole will not be altered. That part of the regression of the nodes which depends on the sun's disturbing force will be greater for the distant satellites than for the near ones; but that which depends on the shape of Jupiter (and which is much more important) will be greater for the near satellites than for the distant ones. On the whole, therefore, the lines of nodes of the interior satellites will regress more rapidly than those of the exterior ones. Their annual regressions (beginning with the second) are, in fact, 12°, 2° 32′, and 41′.

(221.) But the disturbing force of one satellite upon the others will be altered by the circumstance of its orbit not coinciding with its fundamental plane; and the orbit remains long enough in nearly the same position to produce a very sensible irregularity. To discover the nature of this, we must observe that the force of one satellite, perpendicular to the orbit of another, depends wholly upon the inclination of the two orbits, so that, upon increasing the inclination, the disturbing force is affected. Suppose now, to fix our ideas, the second satellite moves in an orbit inclined to its fundamental plane; what is the kind of disturbance that it will produce in the latitude of the first satellite? First, it must be observed, that when moving in the fundamental planes, the forces depending upon the inclination of those planes were taken into account in determining the position of those planes; so that here we have to consider only the alteration produced by the alteration in the second satellite's place. Next, we shall proceed in the same manner as in several preceding instances, by finding what is the motion of the first satellite, related to the motion of the second satellite, which can exist permanently with this inclination of the second satellite. Now, in whatever part the actual orbit of the second is higher above, or less depressed below, the orbit of the first, than the fundamental plane of the second was, at that part there will be a greater force drawing the first satellite up, or a smaller force drawing it down, (in the conjunctions at that part,) than before. The alteration of force, then, will be generally represented by supposing a force to act on the first satellite, at different points of its orbit, towards the same side of its orbit as the side on which the second satellite's orbit is there removed from its fundamental plane, and proportional to the magnitude of that removal. Now, conceiving the inequality introduced into the motion of the first satellite to be a small inclination of its orbit to its fundamental plane, (which is the only inequality of Jupiter's satellites that we consider,) the nodes of this orbit cannot correspond to the places where the second satellite is furthest from its fundamental plane; for then, at one node of the first satellite, the disturbing force, before and after passing that node, being great, and not changing its direction, would not alter the place of the node, but would greatly alter the inclination: and at the opposite node, the force acting in the opposite direction would produce the same effect; and thus the permanency of the inequality would be destroyed. We must then suppose the nodes of the orbit of the first satellite on its fundamental plane to coincide with those of the orbit of the second satellite on its fundamental plane. But is the inclination to be the same way, or the opposite way? To answer this, we must consider that the action of Jupiter's shape would tend to make the nodes of the first satellite regress much more rapidly than those of the second; but as our orbit of the first satellite is assumed to accompany the second in its revolution, the disturbing force depending on the second must be such as to destroy a part of this regression, or to produce (separately) a progression of the nodes of the first; consequently, the disturbing force produced by the second must tend to draw the first from its fundamental plane. (193.) But the disturbing force produced by the second is in the same direction as the distance of the second from the fundamental plane of the second; consequently, the orbit of the first must lie in the same position, with regard to the fundamental plane of the first, in which the orbit of the second lies with regard to the fundamental plane of the second. The same reasoning applies to every other case of an interior satellite disturbed by an exterior; and thus we have the conclusion: If the orbit of one of Jupiter's satellites is inclined to its fundamental plane, it affects the orbit of each of the satellites interior to it with an inclination of the same kind, and with the same nodes.

(222.) Let us now inquire what will be the nature of the inequality produced in the latitude of the third satellite. The same reasoning and the same words may, in every part, be adopted, except that the regression of the nodes of the third satellite, as produced by Jupiter's shape, will be slower than that of the second satellite, and therefore the disturbing force which acts on the third, must now be such as to quicken the regression of its nodes, and must therefore be directed towards its fundamental plane. From this consideration we find, as a general conclusion, if the orbit of one of Jupiter's satellites is inclined to its fundamental plane, it affects the orbit of each of the satellites exterior to it, with an inclination of the opposite kind, but with the same nodes.

(223.) The first satellite's orbit appears to have no sensible inclination to its fundamental plane; but those of the second, third, and fourth are inclined to their fundamental planes, (the second 25′, and the third and fourth about 12,′) and these are found to produce in the others inequalities such as we have investigated.

(224.) It is only necessary to add, that the disturbance of the first satellite by the second produces an alteration in the action of the first on the second, tending to draw the second from its fundamental plane, and therefore to diminish, by a small quantity, the regression of its nodes. In the same manner, the altered action of the third on the second tends to draw the second towards its fundamental plane, and therefore to increase, by a small quantity, the regression of its nodes. There is exactly the same kind of complication with regard to the disturbances of those bodies in latitude as with regard to those in longitude, explained in (150.), &c.

(225.) The only other inequality in latitude, which is sensible, is that depending on the position of the sun, with regard to the nodes of the orbits on the plane of Jupiter's orbit, (that is, with regard to the node of Jupiter's equator on Jupiter's orbit,) and this amounts to only a few seconds. It is exactly analogous to that of the moon, explained in (205.).

Section IX.—*Effects of the Oblateness of Planets upon the Motions of their Satellites.*

(226.) In the investigations of motion about a central body, we have supposed that central body to be a spherical ball. This makes the investigation remarkably simple; for it is demonstrated by mathematicians, that the spherical form possesses the following property: the attraction of all the matter in a sphere upon another body at any distance external to it is exactly the same as if all the matter of the sphere were collected at the centre of the sphere. In the investigation of motion about a centre, we may therefore lay aside (as we have usually done) all consideration of the size of the attracting body, if that body is spherical.

(227.) But the planets are not spherical. Whether or not they have ever been fluid, still they

they have (at least, the earth has) a great extent of fluid on its surface, and the form of this fluid will be affected by the rotation of the planet. The fluid will spread out most where the whirling motion is most rapid, that is, at the equator. Thus it appears from theory, and it is also found from measures, that the earth is not a sphere, but a spheroid, flattened at the north and south poles, and protuberant at the equator. The proportion of the axes differs little from the proportion of 299 : 300; so that a line drawn through the earth's centre, and passing through the equator, is longer than one passing through the poles, by 27 miles.

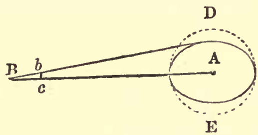
(228.) The flattening of Jupiter is still more remarkable. The proportion of his axes differs little from that of 13 : 14, and thus the difference of his diameters is nearly 6000 miles. In fact, the eye is immediately caught by the elliptic appearance of Jupiter, on viewing him for a moment in a telescope.

(229.) It is our business, in the present section, to point out the general effects of this shape upon the motion of satellites. The agreement of observation with calculation on this point is certainly one of the most striking proofs of the correctness of the theory, "that every particle of matter attracts every other particle, according to the law of Universal Gravitation."

(230.) We will begin with explaining the law according to which an oblate planet attracts a satellite in the plane of its equator.

The spheroid represented by the dark line in *fig.* 48 may be supposed to be formed from the sphere represented by the dotted line, by cutting off a quantity of matter from each pole. To simplify our conception, let us suppose that all the

Fig. 48.



matter cut off was in one lump at each pole; that is, at the points D and E. The attraction of the whole sphere on the satellite B, as we have remarked, is the same as if all the matter of the sphere were collected at A. But the attraction of the part cut off is not the same as if it were collected at A, inasmuch as its distance from B is greater, and as the direction of the attraction to D, or to E, is not the same as that to A. Thus, suppose A D is called 1, and A B is called 10. Since the forces are inversely as the squares of the distances at which the attracting mass is situate, the attraction of the lump D, if at the point A, where its distance from B is 10, may be called $\frac{1}{100}$; but if at D, it must be called $\frac{1}{101}$, since the square of B D is equal to the sum of the squares of B A and A D, that is, to the sum of 100 and 1. Also the direction of attraction is not the same; for, if the attraction of D should draw the satellite through B *b*, and if *b c* be drawn perpendicular to A B, the only effective approach to A is the distance B *c*, which is less than B *b* in the proportion of B A to B D, or of 10 to $\sqrt{101}$; and, therefore, the effective attraction of D, estimated by the space through which it draws the satellite towards A, must be called $\frac{10}{101 \times \sqrt{101}}$. And this is the whole effect which its attraction produces; for though the attraction of D alone tends to draw the satellite above A B, yet the attraction of E will tend to draw it as much below A B; and thus the parts of the force which act perpendicular to A B will destroy each other. We have, then: the attraction of the lump D, if placed at A, would be represented by $\frac{1}{100} = 0{\cdot}01$; but as placed at D, its effective attraction is represented by $\frac{10}{101 \times \sqrt{101}} = 0{\cdot}0098518$. The difference is $0{\cdot}0001482$, or nearly $\frac{1500}{100000}$th of the whole attraction of D, and the same for E. Consequently, the lumps at D and E produce a smaller effective attraction on B than if they were collected at A; but the whole sphere produces the same effect as if its whole mass were collected at A; and, therefore, the part left after cutting away the lumps at D and E produces a greater attraction than if its whole mass were collected at A.

(231.) But it is important to inquire, whether this attraction is greater than if the matter of the spheroid were collected at the centre, in the same proportion at all distances of the satellite. For this purpose, suppose the distance of the satellite to be 20. The same reasoning would show, that the attraction of the lump D, if placed at A, must now be represented by $\frac{1}{400} = 0{\cdot}0025$; but that, if placed at D, its effective attraction is represented by $\frac{20}{401 \times \sqrt{401}} = 0{\cdot}002490653$. The difference now is $0{\cdot}000009347$, or nearly $\frac{375}{100000}$ of the whole attraction of D. Consequently by removing the satellite to twice the distance from A, the difference between the effective attraction of the lump at A and at D, bears to the whole attraction of the lump at A, a proportion four times smaller than before. And, therefore, the attraction of the spheroid, though still greater than if its whole matter were collected at A, differs from that by a quantity, whose proportion to the whole attraction is only one-fourth of what it was before. If we tried different distances in the same manner, we should find, as a general law, that the proportion which the difference (of the actual attraction, and the attraction supposing all the matter collected at the centre) bears to the latter, diminishes as the square of the distance from A increases.

(232.) The attraction of an oblate spheroid upon a satellite, or other body, in the plane of its equator, may, therefore, be stated thus:—There is the same force as if all the matter of the spheroid were collected at its centre, and, besides this, there is an additional force, depending upon the oblateness, whose proportion to the other force diminishes as the square of the distance of the satellite is increased.

(233.) Now, let us investigate the law according-ing to which an oblate spheroid attracts a body, situate in the direction of its axis.

Proceeding in the same manner as before, and supposing the distance A B to be 10, the attraction of the lump, which at A would be represented by $\frac{1}{100}$, will at D be represented by $\frac{1}{81}$, and will at E be represented by $\frac{1}{121}$, (since the distances of D and E from B are respectively 9 and 11.) Hence, if the two equal lumps, D and E, were collected

Fig. 49.

at the centre, their attraction on B would be $\frac{1}{100} + \frac{1}{100} = \frac{1}{50} = 0{\cdot}02$. In the positions D and E, the sum of their attractions on B is $\frac{1}{81} + \frac{1}{121} = 0{\cdot}0206100$. The difference is $0{\cdot}0006100$, by which the attraction in the latter case is the greater. Consequently, the attraction of the lumps in the positions D and E is greater than if they were collected at the centre by nearly $\frac{3}{100}$th of their whole attraction; but the attraction of the whole sphere is the same as if all the matter of the sphere were collected at the centre; therefore, when these parts are removed, they must leave a mass, whose attraction is less than if its whole matter were collected in the centre. With regard to the alteration depending on the distance of B, it would be found, on trial, to follow the same law as before.

(234.) The attraction of a spheroid on a body in the direction of its axis may, therefore, be represented, by supposing the whole matter collected at the centre, and then supposing the attraction to be diminished by a force depending on the oblateness, whose proportion to the whole force diminishes as the square of the distance of the body is increased.

(235.) Since the attraction on a body, in the plane of the equator, is greater than if the mass of the spheroid were collected at its centre, and the attraction on a body in the direction of the axis is less, it will readily be understood, that in taking directions, successively more and more inclined to the equator, on both sides, the attraction successively diminishes. And there is one inclination, at which the attraction is exactly the same as if the whole mass of the spheroid were collected at its centre.

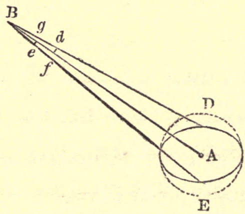
(236.) Now, suppose that a satellite revolves in an orbit, which coincides with the plane of the equator, or makes a small angle with it; what will be the nature of its orbit? For this investigation we have only to consider, that there is acting upon the satellite a force, the same as if all the matter of the spheroid were collected at its centre, and, consequently, proportional inversely to the square of the distance, and that, with this force only, the satellite would move in an ellipse, whose focus coincided with the centre of the spheroid. But besides this, there is a force always directed to the centre, depending on the oblateness. One effect of it will be, that the periodic time will be shorter with the same mean distance, or the mean distance greater with the same periodic time, than if the former were the only force. (46.) Another effect will be, that when the satellite is at its greatest distance, this force will cause the line of apses to regress, and when at its smallest distance, this force will cause the line of apses to progress. (50.) and (53.). If this force, at different distances, were in the same proportion as the other attractive force, it would, on the whole, cause no alteration in the position of the line of apses, (for it would amount to the same as increasing the central mass in a certain proportion, in which case an ellipse, with invariable line of apses, would be described; that is, the regression at the greatest distance would be equal to the progression at the least distance. *See the note to* (98.)). But (231.) the proportion of this force to the other diminishes as the distance is increased. Consequently, the regression at the greatest distance is less than the progression at the least distance, and, therefore, on the whole, the line of apses progresses. Also, the nearer the satellite is to the planet, the greater is the proportion of this force to the other attraction; and, therefore, the more rapid is the progression of the line of apses at every revolution. The progression of the line of apses of the moon's orbit, produced by the earth's oblateness, is so small in comparison with that produced by the sun's disturbing force, that it can hardly be discovered; but the progression of the lines of apses in the orbits of Jupiter's satellites, produced by the oblateness of Jupiter, is so rapid, especially for the nearest satellites, that the part produced by the sun's disturbing force is small in comparison with it.

(237.) We shall now proceed with the investigation of the disturbance in a satellite's latitude, produced by the oblateness of a planet.

(238.) First, It is evident that if the satellite's orbit coincides with the plane of the planet's equator, there will be no force tending to pull it up or down from that plane; and, therefore, it will continue to revolve in that plane. In this case, then, there is no disturbance in latitude; we must, therefore, in the following investigation, suppose the orbit inclined to the plane of the equator.

In *fig.* 50., then, let us consider (as before) the effect of the attractions of the two lumps at D and E, in pulling the satellite B perpendicularly to the line A B. Now D is nearer to B than E is; also the line D B is more inclined than E B to A B.

Fig. 50.



If the attraction of D alone acted, it would in a certain time draw the satellite to *d*; and *fd* would be the part of the motion of B, which is perpendicular to A B; and this motion is upwards. In like manner, if the attraction of E alone drew B to *e* in the same time, *g e* would be the motion perpendicular to A B, and this motion is downwards. When both attractions act, these effects are combined; the question then is, which is greater, *f d* or *g e*? Now, since D is nearer than E, the attraction of D is greater than that of E, therefore B *d* is greater than B *e*; also B *d* is more inclined than B *e* to B A; therefore *d f* is much greater than *g e*. Hence, the force which tends to draw B upwards is the preponderating force; and therefore, on the whole, the combined attractions of D and E will tend to draw the satellite upwards from the line B A. But the attraction of the whole sphere would tend to draw it along the line B A. Therefore, when D and E are removed, the attraction of the remaining mass (that is, the oblate spheroid) will tend to draw B below the line B A. In estimating the attraction of an oblate spheroid, therefore, we must consider, that besides the force directed to the centre of the spheroid, there is always a force perpendicular to the radius vector directed towards the plane of the equator, or tending to draw a satellite from the plane of its orbit towards the plane of the planet's equator. If the satellite is near to the planet, the inequality of the proportion of the distances D B and E B is increased, and the inequality of the inclinations to B A is also increased; and the disturbance is, therefore, much greater for a near satellite than for a distant one.

(239.) We have seen (215.) the effect of this disturbing force in determining the fundamental planes of the orbits of Jupiter's satellites. And from (192.), &c., we can infer, at once, that this force will cause the line of nodes to regress, if the orbit is inclined to the fundamental plane, and the more rapidly as the satellite is nearer to the planet. If there were no other disturbing force, the inclination of those orbits to the plane of Jupiter's equator would be invariable, and their nodes would regress with different velocities, those of the near satellites regressing the quicker. In point of fact, the circumstances of the inner satellites are very nearly the same as if no other disturbing force existed, so great is the effect produced by Jupiter's oblateness.

(240.) The figure of Saturn, including in our consideration the ring which surrounds him, is different from that of Jupiter; but the same principles will apply to the general explanation of its effects on the motion of its satellites. The body of Saturn is oblate, and the forces which it produces are exactly similar to those produced by Jupiter. The effect of the ring may be thus conceived :—If we inscribe a spherical surface in an oblate spheroid, touching its surface at the two poles, the spheroid will be divided into two parts; a sphere whose attraction is the same as if all its matter were collected at its centre, and an equatorial protuberance analogous in form to a ring. The whole irregularity in the attraction of the spheroid is evidently due to the attraction of this ring-like protuberance, since there is no such irregularity in the attraction of the sphere. We infer, therefore, that the irregularity in the attraction of a ring is of the same kind as the irregularity in the attraction of a spheroid, but that it bears a much greater proportion to the whole attraction for the ring than for the spheroid, since the ring produces all the irregularity without the whole attraction. Now, the plane of Saturn's ring coincides with the plane of Saturn's equator, so that the effect of the body and ring together is found by simply adding effects of the same kind, and is the same as if Saturn were very oblate. The rate of progression of the perisaturnium of any satellite, and the rate of regression of its node, will, therefore, be rapid. In other respects it is probable, that the theory of these satellites would be very simple, since all (except the sixth) appear to be very small, and the sun's disturbing force is too small to produce any sensible effects.

(241.) The satellites of Saturn, except the sixth, have been observed so little, that no materials exist upon which a theory can be founded. A careful series of observations on the sixth satellite has lately been made by Bessel, from which, by comparing the observed progress of the perisaturnium and regression of the node, with those calculated on an assumed mass of the ring, the real mass of the ring has been found. It appears, thus, that the mass of the ring (supposing the whole effect due to the ring) is about $\frac{1}{118}$th of the mass of the planet.

(242.) The effect of the earth's oblateness in increasing the rapidity of regression of the moon's nodes is so small, that it cannot be discovered from observation. But the effect on the position of the fundamental plane is discoverable. We have seen (204.) that the moon's line of nodes regresses completely round in $19\frac{1}{2}$ years. The plane of the earth's equator is inclined $23\frac{1}{2}°$ to the earth's orbit, and the line of intersection alters very slowly. At some time, therefore, the line of nodes coincides with the intersection of the plane of the earth's equator and the plane of the earth's orbit, so that the plane of the moon's orbit lies between those two planes; and $9\frac{3}{4}$ years later, the line of nodes again coincides with the same line, but the orbit is inclined the other way, so that the plane of the moon's orbit is more inclined than the plane of the earth's orbit to the plane of the earth's equator. Now it is found, that in the former case the inclination of the moon's orbit to the earth's orbit is greater than in the latter by about $16''$, and this shows, that the plane to which the inclination has been uniform, is neither the plane of the earth's equator, nor that of the earth's orbit, but makes with the latter an angle of about $8''$, and is inclined towards the former.

(243.) There is another effect of the earth's oblateness (the only other effect on the moon which is sensible) that deserves notice. The inclination of the moon's orbit to the earth's orbit is less than $5°$, and the inclination of the earth's equator to the earth's orbit is $23\frac{1}{2}°$. Consequently, when the moon's orbit lies between these two planes, the inclination of the moon's orbit to the earth's equator is about 19°; and when the line of nodes is again in the same position, but the orbit is inclined the other way, the inclination of the moon's orbit to the earth's equator is about 28°. At the latter time, therefore, in consequence of the earth's oblateness, the moon, when farthest from its node, will, by (235.), experience a smaller attraction to the earth than at the former time when farthest from its node. When in the line of nodes, the attractions in the two cases will be equal. On the whole, therefore, the attraction to the earth will be less at the latter time than at the former. For the period of $9\frac{3}{4}$ years, therefore, the earth's attraction on the moon is gradually diminished, and then is gradually increased for the same time. The moon's orbit (47.) becomes gradually larger in the first of these times, and smaller in the second. The change is very minute, but, as explained in (49.), the alteration in the longitude may be sensible. It is found by observation to amount to about 8″, by which the moon is sometimes before her mean place, and sometimes behind it. If the earth's flattening at each pole were more or less than $\frac{1}{300}$th of the semi-diameter, the effects on the moon, both in altering the position of the fundamental plane, and in producing this inequality in the longitude, would be greater or less than the quantities that we have mentioned; and thus we are led to the very remarkable conclusion, that by observing the moon we can discover the amount of the earth's oblateness, supposing the theory to be true. This has been done; and the agreement of the result thus obtained, with that obtained from direct measures of the earth, is one of the most striking proofs of the correctness of the Theory of Universal Gravitation.

THE END.

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6